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Asymptotic Behaviour of Orthogonal Polynomials on the Unit Circle with Asymptotically Periodic Reflection Coefficients

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Let $\{a_n\}_{n \in \mathbb{N}_0}$ with $a_n \in \mathbb{C}$, $a_{n+N} = a_n$ and $|a_n| < 1$ for all $n \in \mathbb{N}_0$, be a periodic sequence of reflection coefficients and let $\{P_n\}_{n \in \mathbb{N}_0}$ be the associated sequence of orthogonal polynomials generated by $P_{n+1} = zP_n - \bar{a}_n P_n^*$. Furthermore let $\{b_n\}_{n \in \mathbb{N}_0}$ be an asymptotically periodic sequence of reflection coefficients which arises by a perturbation of the sequence $\{a_n\}_{n \in \mathbb{N}_0}$ and thus satisfies the conditions $\lim_{\nu \to \infty} b_{j+\nu N} = a_j$ for $j = 0, ..., N-1$, and $|b_n| < 1$ for all $n \in \mathbb{N}_0$. Let $\{\tilde{P}_n\}_{n \in \mathbb{N}_0}$ generated by $\tilde{P}_{n+1} = z\tilde{P}_n - \bar{b}_n \tilde{P}_n^*$ be the disturbed orthogonal polynomials. Using the "periodic" polynomials $\{P_n\}_{n \in \mathbb{N}_0}$ as a comparison system we derive so-called comparative asymptotics for the disturbed polynomials on and off the support of the disturbed orthogonality measure, which consists essentially of several arcs of the unit circle. As a by-product of these results we obtain asymptotically a description of the location of the zeros of $\{\tilde{P}_n\}_{n \in \mathbb{N}_0}$. Finally, a representation for the absolutely continuous part of the disturbed orthogonality measure is derived, and it is shown that there are at most finitely many point measures if the b_n's converge geometrically fast to the a_n's. © 1997 Academic Press

1. INTRODUCTION AND NOTATION

Let $\{P_n\}_{n \in \mathbb{N}_0}$ be a sequence of monic polynomials of degree $\partial P_n = n$ generated by a recurrence relation of the form

$$P_{n+1}(z) = zP_n(z) - \bar{a}_n P_n^*(z), \qquad n \in \mathbb{N}_0, \quad P_0(z) = 1, \tag{1.1}$$

where the *reflection coefficients* or *Schur-parameters* $\{a_n\}_{n \in \mathbb{N}_0}$ satisfy

$$a_n \in \mathbb{C} \qquad \text{and} \qquad |a_n| < 1. \tag{1.2}$$

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In (1.1), P_n^* denotes the reciprocal polynomial of P_n defined by $P_n^*(z)=z^n\bar{P}_n(1/z)$.

It is well known (see e.g. [7, Sections 11ff.]) that because of (1.2) there exists a distribution function σ, i.e., a real bounded nondecreasing function with an infinite set of points of increase, with respect to which the P_n's are orthogonal polynomials, that is,

$$\int_0^{2\pi} e^{-ij\varphi}P_n(e^{i\varphi})\,d\sigma(\varphi)=0 \qquad \text{for} \quad j=0, ..., n-1. \tag{1.3}$$

The function

$$F(z):=\frac{1}{2\pi c_0}\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\sigma(\varphi), \qquad |z|<1, \quad F(0)=1, \tag{1.4}$$

where $c_0:=1/2\pi\int_0^{2\pi}d\sigma(\varphi)\in\mathbb{R}\backslash\{0\}$, is analytic and pseudopositive (i.e., $\operatorname{Re}F(z)>0$) on $|z|<1$ and is called Carathéodory-function (abbreviated by C-function; see e.g. again [7, Section 11]).

Further let us define the sequence of the so-called polynomials of the second kind $\{\Omega_n\}_{n\in\mathbb{N}_0}$ by

$$\Omega_{n+1}(z)=z\Omega_n(z)+\bar{a}_n\Omega_n^*(z), \qquad n\in\mathbb{N}_0, \quad \Omega_0(z)=1. \tag{1.5}$$

The polynomials P_n and Ω_n are related by (cf. [7, p. 7])

$$P_n^*(z)\,\Omega_n(z)+P_n(z)\,\Omega_n^*(z)=2d_nz^n, \quad n\in\mathbb{N}_0,$$
$$\text{where} \quad d_n:=\prod_{j=0}^{n-1}(1-|a_j|^2). \tag{1.6}$$

Finally, let $m\in\mathbb{N}_0$ and let

$$\begin{aligned} P_{n+1}^{(m)}(z)&:=zP_n^{(m)}(z)-\bar{a}_{n+m}P_n^{(m)*}(z), \qquad n\in\mathbb{N}_0, \quad P_0^{(m)}(z):=1\\ \Omega_{n+1}^{(m)}(z)&:=z\Omega_n^{(m)}(z)+\bar{a}_{n+m}\Omega_n^{(m)*}(z), \qquad n\in\mathbb{N}_0, \quad \Omega_0^{(m)}(z):=1 \end{aligned} \tag{1.7}$$

be the mth associated polynomials, investigated by the first author in [15].

In this paper we first study the asymptotic behaviour of polynomials generated by (1.1), when the reflection coefficients $\{a\}_{n\in\mathbb{N}_0}$ are periodic, i.e., if $a_{n+N}=a_n$ for all $n\geqslant n_0$, $n_0\in\mathbb{N}_0$, where $N\in\mathbb{N}$ is fixed. These polynomials generated by a periodic sequence of reflection coefficients will be used then as a "comparison system" for polynomials with asymptotically periodic reflection coefficients $\{b_n\}_{n\in\mathbb{N}_0}$, i.e.,

$$\lim_{\nu\to\infty} b_{n_0+j+\nu N}=a_{n_0+j} \qquad \text{for} \quad j=0, ..., N-1, \tag{1.8}$$

and, under the assumption that $\sum_{n=0}^{\infty} |a_n - b_n| < \infty$, comparative asymptotics are derived in Section 3. As a byproduct we obtain the asymptotic behaviour of the zeros of P_n and $\tilde{P}_n$. Finally, in Section 4 the orthogonality measure $\tilde{\sigma}$ of the disturbed polynomials is investigated. More precisely, a representation of the absolutely continuous part is given and it is shown that $\tilde{\sigma}$ has at most a finite number of mass-points on $[0, 2\pi] \backslash \operatorname{Int}(E_l)$ and no mass-point on $\operatorname{Int}(E_l)$ if $|a_n - b_n| = \mathcal{O}(r^n)$, $0 < r < 1$. For the definition of the set E_l see Section 2.

In contrast to the so-called Szegö case, when the reflection coefficients $\{b_n\}_{n \in \mathbb{N}_0}$ of the perturbed orthogonal polynomials satisfy $\sum_{n=0}^{\infty} |b_n|^2 < \infty$ and the comparison system is $P_n(z) = z^n$, which has been investigated in detail, see e.g. [7, 19], not much is known about orthogonal polynomials and perturbations of these polynomials if they are beyond the Szegö-class. An exceptional case are polynomials orthogonal on an arc, which have been investigated by Geronimus [4] and Akhiezer [1], and perturbations of polynomials with constant reflection coefficients, which were studied very recently by Golinskii, *et al.* [9] at about the same time as the main results of this paper were obtained. Our approach is completely different from that one in [9] and is heavily based on our results on orthogonal polynomials with periodic reflection coefficients [18] and on comparative asymptotics [16].

2. ASYMPTOTIC PROPERTIES OF ORTHOGONAL POLYNOMIALS WITH PERIODIC REFLECTION COEFFICIENTS

In the following we denote by $\{P_n\}_{n \in \mathbb{N}_0}$ a sequence of polynomials which satisfies (1.1), (1.2) and the additional condition that the reflection coefficients are periodic from a certain index onward, i.e.,

$$a_{n+N} = a_n, \qquad \text{for all} \quad n \geqslant n_0, \quad n_0 \in \mathbb{N}_0, \quad N \in \mathbb{N}. \tag{2.1}$$

From the identity (see [15, Corollary 3.1])

$$2P_{n+n_0} = (P_{n_0} + P_{n_0}^*)\, P_n^{(n_0)} + (P_{n_0} - P_{n_0}^*)\, \Omega_n^{(n_0)}, \qquad n \in \mathbb{N}_0, \tag{2.2}$$

the asymptotic behaviour of the P_n's can be obtained from the asymptotic behaviour of the n_0th associated polynomials $P_n^{(n_0)}$ and $\Omega_n^{(n_0)}$, where now the reflection coefficients $\{a_n^{(n_0)}\}_{n \in \mathbb{N}_0}$ of the $P_n^{(n_0)}$'s are purely periodic, i.e., $a_{n+N}^{(n_0)} = a_n^{(n_0)}$ for all $n \in \mathbb{N}_0$. Thus it suffices to study the asymptotic behaviour of orthogonal polynomials with purely periodic reflection coefficients, i.e.,

$$n_0 = 0 \qquad \text{in (2.1)}. \tag{2.3}$$

Throughout this paper we make the following

Assumption 2.1. We exclude the case $a_n = 0$ for $n \geqslant n_0 \in \mathbb{N}_0$, because then the orthogonal polynomials are the so-called Bernstein–Szegö polynomials (cf. [2; 19, p. 31]) and are of the form $P_{n_0+j}(z) = z^j P_{n_0}(z)$, $j \in \mathbb{N}_0$, thus the asymptotic behaviour is obvious. A perturbation of these reflection coefficients leads because of the assumption (3.12) below to the well known Szegö-theory (see e.g. [8; 19]).

In [18] we studied the orthogonality measures generated by periodic reflection coefficients and properties of the corresponding orthogonal polynomials. Let us give a short summary of those results, which are important in what follows: Let $\{P_n\}_{n \in \mathbb{N}_0}$ be a sequence of orthogonal polynomials generated by the periodic reflection coefficients $\{a_n\}_{n \in \mathbb{N}_0}$ with $a_{n+N} = a_n$ and $|a_n| < 1$ for $n \in \mathbb{N}_0$. Then the measure σ to which the polynomials P_n, $n \in \mathbb{N}_0$, are orthogonal can be described as follows. There exist polynomials R, W, and A with the properties, $l \leqslant N$,

$$R(z) = c_R \prod_{j=1}^{2l} (z - e^{i\varphi_j}), \qquad R = R^*,$$

and

$$\mathscr{R}(\varphi) := e^{-il\varphi} R(e^{i\varphi}) \leqslant 0 \quad \text{on } E_l,$$

where the zeros of R satisfy $\varphi_1 < \cdots < \varphi_{2l}$ and $\varphi_{2l} - \varphi_1 < 2\pi$, where $c_R \in \mathbb{C}$, and where

$$E_l := \bigcup_{j=1}^{l} [\varphi_{2j-1}, \varphi_{2j}] \qquad \text{resp.} \qquad \Gamma_{E_l} := \{e^{i\varphi} : \varphi \in E_l\}.$$

W is a divisor of R and is self-reciprocal, i.e.,

$$R = VW \qquad \text{and} \qquad W = W^*,$$

and A is a self-reciprocal polynomial of degree $l - \partial V$ which has no zero on Γ_{E_l} and an odd number of zeros on each arc $\{e^{i\varphi} : \varphi \in (\varphi_{2j}, \varphi_{2j+1})\}$, i.e.,

$$A(z) = c_A \prod_{j=1}^{p} (z - e^{i\xi_j}) \prod_{j=p+1}^{m^*} (z - z_j)^{m_j} \qquad \text{and} \qquad A = -A^*, \tag{2.4}$$

where $c_A \in \mathbb{C}$, $p + \sum_{p+1}^{m^*} m_j = l - \partial V$, $\xi_1, ..., \xi_p \in [0, 2\pi] \backslash E_l$ pairwise distinct, $z_j \notin \Gamma_{E_l} \cup \{0\}$, $j = p+1, ..., m^*$, and $\{z_{p+1}, ..., z_{m^*}\} = \{1/\overline{z_{p+1}}, ..., 1/\overline{z_{m^*}}\}$.

Then the measure σ with respect to which the P_n's are orthogonal can be represented in the form

$$d\sigma(\varphi) =: d\sigma(\varphi; A, W) = f(\varphi; A, W)\, d\varphi - 2\pi \sum_{j=1}^{p} \mu_j e^{-i\varphi}\, \delta(e^{i\varphi} - e^{i\xi_j})\, d\varphi, \tag{2.5}$$

where the absolute continuous part $f(\varphi; A, W)$ of $\sigma(\varphi)$ is of the form

$$f(\varphi; A, W) = \begin{cases} \left| \dfrac{W(e^{i\varphi})}{A(e^{i\varphi}) \sqrt{R(e^{i\varphi})}} \right|, & \varphi \in E_l \\ 0, & \text{otherwise,} \end{cases} \tag{2.6}$$

and where $\delta(\cdot - e^{i\xi_j})$ denotes the Dirac measure at the point $e^{i\xi_j}$,

$$\mu_j := \frac{W(e^{i\xi_j})}{A_j(e^{i\xi_j}) \sqrt{R(e^{i\xi_j})}} \qquad \text{and} \qquad A_j(z) = A(z)/(z - e^{i\xi_j}).$$

Here and in what follows we always choose that branch of $\sqrt{R}$ which is analytic on $\mathbb{C} \backslash \Gamma_{E_l}$ and satisfies (compare [17, (2.1)])

$$\begin{gathered} \operatorname{sgn} \sqrt{R(e^{i\varphi})} = (-1)^j\, e^{i \frac{l}{2} \varphi}, \\ \varphi \in (\varphi_{2j}, \varphi_{2j+1}), \quad j = 1, ..., l, \quad \varphi_{2l+1} := \varphi_1 + 2\pi. \end{gathered} \tag{2.7}$$

It is crucial in what follows that the C-function $F(\cdot\,; A, W)$, associated by (1.4) with the distribution $\sigma(\varphi; A, W)$, exists not only on $|z| < 1$ but also on $\mathbb{C} \backslash (\Gamma_{E_l} \cup \{e^{i\xi_1}, ..., e^{i\xi_p}\})$ and has the representation

$$F(z; A, W) = \frac{V(z)\, B(z) + \sqrt{R(z)}}{V(z)\, A(z)}, \qquad z \in \mathbb{C} \backslash (\Gamma_{E_l} \cup \{e^{i\xi_1}, ..., e^{i\xi_p}\}), \tag{2.8}$$

where $B := B(\cdot\,; A, W)$ is that uniquely determined self-reciprocal polynomial $B = B^*$ of degree $\partial B = \partial A$, which interpolates the rational function $W/\sqrt{R}$ at the zeros $e^{i\xi_1}, ..., e^{i\xi_p}$ of A and $-W/\sqrt{R}$ at the zeros $z_{p+1}, ..., z_{m^*}$ of A, i.e.,

$$B(e^{i\xi_j}) = \frac{W(e^{i\xi_j})}{\sqrt{R(e^{i\xi_j})}}, \qquad j = 1, ..., p$$

$$B^{(\nu)}(z_j) = -\left(\frac{W}{\sqrt{R}} \right)^{(\nu)} (z_j), \qquad j = p+1, ..., m^*, \quad \nu = 0, ..., m_j - 1.$$

Thus $F(\cdot\,; A, W)$ is analytic on $\mathbb{C} \backslash (\Gamma_{E_l} \cup \{e^{i\xi_1}, ..., e^{i\xi_p}\})$ and has simple poles at the points $z = e^{i\xi_j}$, $j = 1, ..., p$.

Further, we showed in [18] that a sequence of (purely) periodic reflection coefficients induces so-called complex Chebyshev-polynomials on E_l (abbreviated T-polynomials) $\mathcal{T}_N(z) = z^N + \cdots$ and $\mathcal{U}_{N-l}(z) = z^{N-l} + \cdots$ of the first and second kind, respectively. These are self-reciprocal polynomials, i.e., $\mathcal{T}_N = \mathcal{T}_N^*$ and $\mathcal{U}_{N-l}^*$, which satisfy the relation

$$\mathcal{T}_N^2(z) - R(z)\,\mathcal{U}_{N-l}^2(z) = L^2 z^N$$
$$\text{with} \quad L = \sqrt{4 \prod_{j=0}^{N-1} (1 - |a_j|^2)} = 2\sqrt{d_N} < 2. \tag{2.9}$$

In particular we obtain for $z = e^{i\varphi}$, $\varphi \in [0, 2\pi]$, that

$$t_N^2(\varphi) - \mathcal{R}(\varphi)\, u_{N-l}^2(\varphi) = L^2, \tag{2.10}$$

where t_N, u_{N-l}, and $\mathcal{R}$ are real trigonometric polynomials given by

$$t_N(\varphi) := e^{-i\frac{N}{2}\varphi} \mathcal{T}_N(e^{i\varphi}), \quad u_{N-l}(\varphi) := e^{-i\frac{N-l}{2}\varphi} \mathcal{U}_{N-l}(e^{i\varphi})$$
$$\mathcal{R}(\varphi) = e^{-il\varphi} R(e^{i\varphi}). \tag{2.11}$$

The complex T-polynomials as well as the polynomials R, A and B can be expressed explicitly in terms of the orthogonal and the nth associated orthogonal polynomials as follows (cf. [18, Theorem 4.3]): For every $n \in \mathbb{N}_0$ there holds

$$R\mathcal{U}_{N-l}^2 = \frac{1}{4}\left[(P_N^{(n)} + \Omega_N^{(n)} + P_N^{(n)*} + \Omega_N^{(n)*})^2 - 16 z^N \prod_{j=n}^{n+N-1} (1 - |a_j|^2) \right] \tag{2.12}$$

$$VA\mathcal{U}_{N-l} = \frac{1}{d_n z^n} (P_n P_{n+N}^* - P_n^* P_{n+N}) = P_N^*(z) - P_N(z) \tag{2.13}$$

$$VB\mathcal{U}_{N-l} = \frac{1}{2d_n z^n} (P_n^* \Omega_{n+N} - \Omega_n^* P_{n+N} - \Omega_n P_{n+N}^* + P_n \Omega_{n+N}^*)$$
$$= \frac{1}{2} [(\Omega_N + \Omega_N^*) - (P_N + P_N^*)] \tag{2.14}$$

$$\mathcal{T}_N = \frac{1}{2} (P_N^{(n)} + \Omega_N^{(n)} + P_N^{(n)*} + \Omega_N^{(n)*}). \tag{2.15}$$

Finally we define for every $n \in \mathbb{N}_0$

$$Q_{n+l}(z) := -V(z)[A(z)\,\Omega_n(z) + B(z)\,P_n(z)], \qquad \partial Q_{n+l} = n + l. \tag{2.16}$$

These polynomials Q_{n+l} together with the P_n's satisfy (cf. [17, Theorem 3.9])

$$\begin{aligned}\sqrt{R(z)}\,P_n(z)-Q_{n+l}(z)&=\dot{\mathcal{O}}(z^n)\\ \sqrt{R(z)}\,P_n^*(z)-Q_{n+l}^*(z)&=\mathcal{O}(z^{n+1})\\ Q_{n+l}^{[*]}(e^{i\xi_j})=-(\sqrt{R}\,P_n)(e^{i\xi_j}), &\qquad j=1,..., p\\ (Q_{n+l}^{[*]})^{(\nu)}(z_j)=(\sqrt{R}\,P_n^{[*]})^{(\nu)}(z_j), &\qquad j=p+1, ..., m^*,\quad \nu=0, ..., m_j-1,\end{aligned}\tag{2.17}$$

where $^{[*]}$ means that the equations are fulfilled both for the polynomials P_n, Q_{n+l} and for their reciprocal polynomials P_n^*, Q_{n+l}^*. Further, by [18, Theorem 4.2] the following relations hold for every $n\in\mathbb{N}_0$ and $\nu\in\mathbb{N}$:

$$\begin{aligned}2P_{n+\nu N}(z)&=P_n(z)\,\mathcal{T}_{\nu N}(z)+Q_{n+l}(z)\,\mathcal{U}_{\nu N-l}(z)\\ 2Q_{(n+\nu N)+l}(z)&=Q_{n+l}(z)\,\mathcal{T}_{\nu N}(z)+R(z)\,P_n(z)\,\mathcal{U}_{\nu N-l}(z);\end{aligned}\tag{2.18}$$

here the polynomials $\mathcal{T}_{\nu N}$ and $\mathcal{U}_{\nu N-l}$ are monic complex T-polynomials on E_l of degree νN and $\nu N-l$, respectively. Thus these polynomials fulfill

$$\mathcal{T}_{\nu N}^2(z)-R(z)\,\mathcal{U}_{\nu N-l}^2(z)=L_{\nu N}^2 z^{\nu N}\qquad\text{with}\quad L_{\nu N}=\sqrt{4\prod_{j=0}^{\nu N-1}(1-|a_j|^2)}\tag{2.19}$$

and they can be explicitly expressed by

$$\begin{aligned}\mathcal{T}_{\nu N}(z)&=\frac{1}{2^{\nu-1}}(Lz^{N/2})^{\nu}\,T_\nu\left(\frac{\mathcal{T}_N(z)}{Lz^{N/2}}\right)=z^{\nu N}+\cdots\\ \mathcal{U}_{\nu N-l}(z)&=\frac{1}{2^{\nu-1}}\,\mathcal{U}_{N-l}(z)(Lz^{N/2})^{\nu-1}\,U_{\nu-1}\left(\frac{\mathcal{T}_N(z)}{Lz^{N/2}}\right)=z^{\nu N-l}+\cdots,\end{aligned}\tag{2.20}$$

where T_ν and $U_{\nu-1}$ denote the classical Chebyshev-polynomials, i.e., $T_\nu(x)=\cos(\nu\arccos x)=2^{\nu-1}x^\nu+\cdots$ and $U_{\nu-1}(x)=\sin(\nu\arccos x)/\sin\arccos x=2^{\nu-1}x^{\nu-1}+\cdots$.

Finally, we define the following finite sets which play an essential part in the asymptotic behaviour of the P_n's:

$$\begin{aligned}\mathcal{N}&:=\{z\in\mathbb{C}\backslash\Gamma_{E_l}:\sqrt{R(z)}\,P_m(z)+Q_{m+l}(z)=0,\ m\in\{0,...,N-1\}\}\\ \mathcal{N}^*&:=\{z\in\mathbb{C}\backslash\Gamma_{E_l}:\sqrt{R(z)}\,P_m^*(z)+Q_{m+l}^*(z)=0,\ m\in\{0,...,N-1\}\}.\end{aligned}\tag{2.21}$$

Let us remark at this point that the set $\mathcal{N}^*$ can be written as $\mathcal{N}^* = \{1/\bar{z}: z \in \mathcal{N} \backslash \{0\}\}$, since $R = R^*$. Further, as we will show in Remark 2.2 below, $\mathcal{N}$ is a subset of the closed unit disk $\{z \in \mathbb{C}: |z| \leqslant 1\}$ and $\mathcal{N} \cap \{z \in \mathbb{C}: |z| = 1\} = \{e^{i\xi_1}, ..., e^{i\xi_p}\}$, i.e., the points from $\mathcal{N}$ with modulus 1 are exactly the mass-points of $\sigma(\varphi; A, W)$ from (2.5).

As it turned out it is more natural to study asymptotics of orthonormal polynomials instead of monic orthogonal polynomials. Thus we denote

$$\Phi_n(z) := \frac{P_n(z)}{\sqrt{d_n}}, \qquad \Psi_n(z) := \frac{\Omega_n(z)}{\sqrt{d_n}}, \qquad \mathcal{Q}_{n+l}(z) := \frac{Q_{n+l}(z)}{\sqrt{d_n}}. \tag{2.22}$$

By $c_0 := 1/2\pi \int_0^{2\pi} d\sigma(\varphi; A, W) = 1$, which follows from (2.8) and (2.12)–(2.14), and by [7, (2.7) and (4.2)] the Φ_n's are orthonormal with respect to $\sigma(\varphi; A, W)$, i.e.,

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{i\varphi}) \overline{\Phi_m(e^{i\varphi})}\, d\sigma(\varphi; A, W) = \delta_{n,m}.$$

Very important for the proofs of our main results is

LEMMA 2.1. *Let $\mathcal{T}_N$, $\mathcal{U}_{N-l}$ be complex T-polynomials on E_l and let L be the constant from* (2.9).

(a) *For every closed* (*not necessarily bounded*) *set $M \subset \mathbb{C} \backslash \Gamma_{E_l}$ there exist constants $\varrho_1 < L$ and $\varrho_2 > L$ such that*

$$\begin{aligned} |\mathcal{T}_N(z) - \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)| &\leqslant \varrho_1 \\ |\mathcal{T}_N(z) + \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)| &\geqslant \varrho_2 \end{aligned} \qquad \textit{uniformly on } M. \tag{2.23}$$

(b) *For $\varphi \in E_l$ let us set*

$$\sqrt[-]{R(e^{i\varphi})} := \lim_{r \to 1^-} \sqrt{R(re^{i\varphi})} \qquad \textit{and} \qquad \sqrt[+]{R(e^{i\varphi})} := \lim_{r \to 1^+} \sqrt{R(re^{i\varphi})}. \tag{2.24}$$

Then $\sqrt[-]{R(e^{i\varphi})} = -\sqrt[+]{R(e^{i\varphi})}$ and

$$\begin{aligned} &|\mathcal{T}_N(e^{i\varphi}) \pm \sqrt[-]{R(e^{i\varphi})}\, \mathcal{U}_{N-l}(e^{i\varphi})| \\ &\quad = |\mathcal{T}_N(e^{i\varphi}) \pm \sqrt[+]{R(e^{i\varphi})}\, \mathcal{U}_{N-l}(e^{i\varphi})| = L. \end{aligned} \tag{2.25}$$

Proof. (a) By applying (2.18) successively ν-times on $\sqrt{R}\, P_{(m+(\nu-1)N)+N} \pm Q_{(m+(\nu-1)N+N)+l}$ and by using (2.1) and the fact that $L = 2\sqrt{d_N}$ we get for all $m \in \{0, ..., N-1\}$ and all $\nu \in \mathbb{N}_0$

$$\sqrt{R(z)}\,\Phi^*_{m+\nu N}(z) \pm \mathcal{Q}^*_{(m+\nu N)+l}(z)$$
$$= \left[\frac{\mathcal{T}_N(z) \pm \sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}{L}\right]^\nu (\sqrt{R(z)}\,\Phi^*_m(z) \pm \mathcal{Q}^*_{m+l}(z)). \tag{2.26}$$

Further we obtain from (2.8) and (2.16) that

$$\sqrt{R(z)}\,\Phi^*_{m+\nu N}(z) - \mathcal{Q}^*_{(m+\nu N)+l}(z)$$
$$= V(z)\,A(z)(\Phi^*_{m+\nu N}(z)\,F(z; A, W) - \Psi^*_{m+\nu N}(z)),$$

where the right-hand side tends to zero as $\nu \to \infty$ on $|z| < 1$ by [14, Theorem 2.1]. This fact together with $\Phi^*_{m+\nu N}(z) \to \infty$ as $\nu \to \infty$ on $|z| < 1$ (compare [7, Theorem 21.1], (2.1), and Assumption 2.1) implies that

$$\sqrt{R(z)}\,\Phi^*_{m+\nu N}(z) + \mathcal{Q}^*_{(m+\nu N)+l}(z) \xrightarrow[\nu \to \infty]{} \infty \qquad \text{on} \quad |z| < 1,$$

i.e., by (2.26)

$$\left[\frac{\mathcal{T}_N(z) + \sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}{L}\right]^\nu (\sqrt{R(z)}\,\Phi^*_m(z) + \mathcal{Q}^*_{m+l}(z)) \xrightarrow[\nu \to \infty]{} \infty \text{ on } |z| < 1. \tag{2.27}$$

The last convergence is only possible if

$$\left|\frac{\mathcal{T}_N(z) + \sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}{L}\right| > 1 \qquad \text{on} \quad |z| < 1. \tag{2.28}$$

Further, by (2.9) we have

$$\left|\frac{\mathcal{T}_N(z) + \sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}{L}\right| \cdot \left|\frac{\mathcal{T}_N(z) - \sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}{z^N L}\right| = 1, \tag{2.29}$$

which implies by (2.28) that

$$\left|\frac{\mathcal{T}_N(z) - \sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}{z^N L}\right| < 1 \qquad \text{on} \quad |z| < 1. \tag{2.30}$$

Since R, $\mathcal{T}_N$, and $\mathcal{U}_{N-l}$ are self-reciprocal polynomials we immediately get an estimate which is valid outside the unit disk. Indeed, let $|z|<1$ and $y:=1/\bar{z}$; then by (2.30),

$$\left|\frac{\mathcal{T}_N(y)-\sqrt{R(y)}\,\mathcal{U}_{N-l}(y)}{L}\right|$$
$$=\left|\frac{\mathcal{T}_N(z)-\sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}{z^N L}\right|<1, \qquad |y|>1. \tag{2.31}$$

Now let $z=e^{i\varphi}$, $\varphi\notin E_l$. Then it follows by (2.9) that $|\mathcal{T}_N(e^{i\varphi})|>L$, $\mathcal{U}_{N-l}(e^{i\varphi})\neq 0$, which gives together with (2.10), note that $\mathcal{R}(\varphi)>0$ for $\varphi\notin E_l$,

$$|\mathcal{T}_N(e^{i\varphi})-\sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})|\neq|\mathcal{T}_N(e^{i\varphi})+\sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})|. \tag{2.32}$$

Now recall that by (2.9),

$$|\mathcal{T}_N(e^{i\varphi})-\sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})|\cdot|\mathcal{T}_N(e^{i\varphi})+\sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})|=L^2$$

and that $\mathcal{T}_N\pm\sqrt{R}\,\mathcal{U}_{N-l}$ is continuous on $\mathbb{C}\setminus\Gamma_{E_l}$; then from (2.30), (2.31), and (2.32) it follows that

$$\left|\frac{\mathcal{T}_N(e^{i\varphi})-\sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})}{L}\right|<1, \qquad \varphi\notin E_l.$$

Summing up we have shown that the first estimate in (2.23) holds true pointwise for every $z\in M$ and by (2.29) and (2.30) the second one as well. The uniform estimates follow from the maximum-principle together with the pointwise estimates and the fact that

$$\lim_{z\to\infty}\frac{\mathcal{T}_N(z)-\sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}{L}=\frac{L}{2}=\sqrt{d_N}<1.$$

To see the last limit relation consider for $|z|>1$ the identity

$$\mathcal{T}_N(z)-\sqrt{R(z)}\,\mathcal{U}_{N-l}(z)$$
$$=\mathcal{T}^*_N(z)-\sqrt{R^*(z)}\,\mathcal{U}^*_{N-l}(z)$$
$$=\overline{\left[\frac{\mathcal{T}_N(y)-\sqrt{R(y)}\,\mathcal{U}_{N-l}(y)}{y^N}\right]} \qquad \text{with} \quad y:=\frac{1}{\bar{z}}$$

and

$$\frac{1}{y^N}(\mathcal{T}_N(y)-\sqrt{R(y)}\,\mathcal{U}_{N-l}(y))\Big|_{y=0}=\frac{L^2}{2}$$

(note that $\mathcal{T}_N(0)+\sqrt{R(0)}\,\mathcal{U}_{N-l}(0)=2$ by (2.12) and (2.15) and recall (2.9)).

Part (b) follows immediately with the help of (2.10) taking into consideration that $\mathcal{R}(\varphi)\leqslant 0$ on E_l and the properties of the square-root $\sqrt{R}$, see (2.7) (compare also [17, Remark 3.2]). ∎

From Lemma 2.1 there immediately follow the first asymptotic relations

COROLLARY 2.1. (a) *Uniformly on every compact subset of* $\mathbb{C}\backslash\Gamma_{E_l}$ *we have*

$$\lim_{n\to\infty}(\sqrt{R(z)}\,\Phi_n(z)-\mathcal{Q}_{n+l}(z))=0$$

and

$$\lim_{n\to\infty}(\sqrt{R(z)}\,\Phi_n^*(z)-\mathcal{Q}_{n+l}^*(z))=0.$$

(b) *Further*

$$\lim_{n\to\infty}\left(\frac{\mathcal{Q}_{n+l}(z)}{\Phi_n(z)}-\sqrt{R(z)}\right)=0 \qquad \textit{for} \quad z\in\mathbb{C}\backslash(\Gamma_{E_l}\cup\mathcal{N})$$

$$\lim_{n\to\infty}\left(\frac{\mathcal{Q}_{n+l}^*(z)}{\Phi_n^*(z)}-\sqrt{R(z)}\right)=0 \qquad \textit{for} \quad z\in\mathbb{C}\backslash(\Gamma_{E_l}\cup\mathcal{N}^*).$$

Again both limit relations hold uniformly compact on $\mathbb{C}\backslash(\Gamma_{E_l}\cup\mathcal{N})$ *resp.* $\mathbb{C}\backslash(\Gamma_{E_l}\cup\mathcal{N}^*)$.

Proof. As in (2.26) there holds for $m\in\{0,...,N-1\}$ and $\nu\in\mathbb{N}_0$,

$$\sqrt{R(z)}\,\Phi_{m+\nu N}(z)\pm\mathcal{Q}_{(m+\nu N)+l}(z)$$
$$=\left[\frac{\mathcal{T}_N(z)\pm\sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}{L}\right]^{\nu}(\sqrt{R(z)}\,\Phi_m(z)\pm\mathcal{Q}_{m+l}(z)). \tag{2.33}$$

Now part (a) immediately follows from (2.26), (2.33), and Lemma 2.1.

(b) Let us write for all $z \in \mathbb{C}\backslash\Gamma_{E_l}$, $m \in \{0, ..., N-1\}$ and $\nu \in \mathbb{N}_0$:

$$\sqrt{R(z)} - \frac{\mathcal{Q}_{(m+\nu N)+l}(z)}{\Phi_{m+\nu N}(z)}$$

$$= \frac{\sqrt{R(z)}\,\Phi_{m+\nu N}(z) - \mathcal{Q}_{(m+\nu N)+l}(z)}{\Phi_{m+\nu N}(z)}$$

$$= \frac{2 \cdot \sqrt{R(z)}(\sqrt{R(z)}\,\Phi_{m+\nu N}(z) - \mathcal{Q}_{(m+\nu N)+l}(z))}{(\sqrt{R(z)}\,\Phi_{m+\nu N}(z) - \mathcal{Q}_{(m+\nu N)+l}(z)) + (\sqrt{R(z)}\,\Phi_{m+\nu N}(z) + \mathcal{Q}_{(m+\nu N)+l}(z))}$$

$$= \frac{2 \cdot \sqrt{R(z)}}{1 + \left(\dfrac{\mathcal{T}_N(z) + \sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}{\mathcal{T}_N(z) - \sqrt{R(z)}\,\mathcal{U}_{N-l}(z)}\right)^{\nu} \dfrac{\sqrt{R(z)}\,\Phi_m(z) + \mathcal{Q}_{m+l}(z)}{\sqrt{R(z)}\,\Phi_m(z) - \mathcal{Q}_{m+l}(z)}},$$

where we have used (2.33) for the last identity. From (2.23) we get

$$\frac{|\mathcal{T}_N(z) + \sqrt{R(z)}\,\mathcal{U}_{N-l}(z)|}{|\mathcal{T}_N(z) - \sqrt{R(z)}\,\mathcal{U}_{N-l}(z)|} > 1 \qquad \text{uniformly compact on } z \in \mathbb{C}\backslash\Gamma_{E_l},$$

and the first assertion follows. The second convergence can be shown in the same way. ▌

As usual let

$$\mathrm{Int}(E_l) := \bigcup_{j=1}^{l} (\varphi_{2j-1}, \varphi_{2j})$$

be the interior of E_l. In what follows let for convenience for $\varphi \in E_l$

$$\sqrt{R(e^{i\varphi})} \text{ denote one of the boundary values } \sqrt[+]{R(e^{i\varphi})} \text{ or } \sqrt[-]{R(e^{i\varphi})} \tag{2.34}$$

defined in (2.24). Then by Lemma 2.1(b),

$$\left|\frac{\mathcal{T}_N(e^{i\varphi}) \pm \sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})}{L}\right| = 1, \qquad \varphi \in E_l,$$

and, since $\mathcal{T}_N$ and $\mathcal{U}_{N-l}$ are self-reciprocal,

$$e^{-i\frac{N}{2}\varphi}\,\frac{\mathcal{T}_N(e^{i\varphi}) + \sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})}{L}$$

$$= \overline{\left[e^{-i\frac{N}{2}\varphi}\,\frac{\mathcal{T}_N(e^{i\varphi}) - \sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})}{L}\right]}, \qquad \varphi \in E_l.$$

Hence, for every $\varphi \in E_l$ there exists a real value $\gamma := \gamma(\varphi)$ such that

$$e^{\pm i\gamma} = e^{-i\frac{N}{2}\varphi} \frac{\mathcal{T}_N(e^{i\varphi}) \pm \sqrt{R(e^{i\varphi})}\, \mathcal{U}_{N-l}(e^{i\varphi})}{L}$$

$$= \frac{t_N(\varphi) \pm i(-1)^j \sqrt{|\mathcal{R}(\varphi)|}\, u_{N-l}(\varphi)}{L}, \qquad \varphi \in [\varphi_{2j-1}, \varphi_{2j}] \qquad (2.35)$$

where the last equation follows by (2.11).

Now we can state the following theorem which gives the asymptotic behaviour of the orthonormal polynomials Φ_n as $n \to \infty$ on the complex plane $\mathbb{C}$.

THEOREM 2.1. *Let $m \in \{0, ..., N-1\}$.*

(a) *Uniformly compact on $\mathbb{C} \backslash \Gamma_{E_l}$ it holds that*

$$\lim_{\nu \to \infty} \left[2\Phi_{m+\nu N}(z) - \left(\frac{\mathcal{T}_N(z) + \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)}{L} \right)^{\nu} \left(\Phi_m(z) + \frac{\mathcal{Q}_{m+l}(z)}{\sqrt{R(z)}} \right) \right] = 0$$

$$\lim_{\nu \to \infty} \left[2\Phi^*_{m+\nu N}(z) - \left(\frac{\mathcal{T}_N(z) + \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)}{L} \right)^{\nu} \left(\Phi^*_m(z) + \frac{\mathcal{Q}^*_{m+l}(z)}{\sqrt{R(z)}} \right) \right] = 0.$$

(b) *Let $\varphi \in E_l$ and $\gamma = \gamma(\varphi)$ be defined as in (2.35). Then for all $\nu \in \mathbb{N}_0$ it holds that*

$$\Phi_{m+\nu N}(e^{i\varphi}) = e^{i\frac{N\nu}{2}\varphi} \left[\Phi_m(e^{i\varphi}) \cos \nu\gamma + \frac{i \sin \nu\gamma}{\sqrt{R(e^{i\varphi})}}\, \mathcal{Q}_{m+l}(e^{i\varphi}) \right]$$

$$\Phi^*_{m+\nu N}(e^{i\varphi}) = e^{i\frac{N\nu}{2}\varphi} \left[\Phi^*_m(e^{i\varphi}) \cos \nu\gamma + \frac{i \sin \nu\gamma}{\sqrt{R(e^{i\varphi})}}\, \mathcal{Q}^*_{m+l}(e^{i\varphi}) \right].$$

Proof. For abbreviation we use the notation $P^{[*]}$, which means that for $P^{[*]}$, P or P^* can be plugged in. Part (a) follows from Lemma 2.1(a) and

$$2\Phi^{[*]}_{m+\nu N}(z) = \left(\Phi^{[*]}_{m+\nu N}(z) + \frac{\mathcal{Q}^{[*]}_{(m+\nu N)+l}(z)}{\sqrt{R(z)}} \right) + \left(\Phi^{[*]}_{m+\nu N}(z) - \frac{\mathcal{Q}^{[*]}_{(m+\nu N)+l}(z)}{\sqrt{R(z)}} \right)$$

$$= \left(\frac{\mathcal{T}_N(z) + \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)}{L} \right)^{\nu} \left(\Phi^{[*]}_m(z) + \frac{\mathcal{Q}^{[*]}_{m+l}(z)}{\sqrt{R(z)}} \right)$$

$$+ \left(\frac{\mathcal{T}_N(z) - \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)}{L} \right)^{\nu} \left(\Phi^{[*]}_m(z) - \frac{\mathcal{Q}^{[*]}_{m+l}(z)}{\sqrt{R(z)}} \right), \qquad (2.36)$$

where we have used (2.26) and (2.33) for the last identity.

(b) From (2.33) and the definition of γ we get for all $\varphi \in \mathrm{Int}(E_l)$

$$\Phi_{m+\nu N}(e^{i\varphi}) \pm \frac{\mathcal{Q}_{(m+\nu N)+l}(e^{i\varphi})}{\sqrt{R(e^{i\varphi})}} = e^{i\frac{N\nu}{2}\varphi} e^{\pm i\nu\gamma} \left(\Phi_m(e^{i\varphi}) \pm \frac{\mathcal{Q}_{m+l}(e^{i\varphi})}{\sqrt{R(e^{i\varphi})}} \right).$$

If we add these equations (where we use first the positive and then the negative sign), we get the representation of $\Phi_{m+\nu N}$ stated in the theorem on the interior of Γ_{E_l}. By reasons of continuity the assertion follows also for the boundary-points of Γ_{E_l} (compare also the proof of Corollary 2.2 below).

In an analog way one shows the representation of $\Phi^*_{m+\nu N}$. ∎

REMARK 2.1. (a) *From Theorem* 2.1 *and Lemma* 2.1 *we see that in contrast to the Szegö-class, both the orthonormal polynomials* Φ_n *and the reciprocal polynomials* Φ_n^* *tend to infinity as* $n \to \infty$ *on* $\mathbb{C} \backslash \Gamma_{E_l}$ *except at the finite points from* $\mathcal{N}$ *resp.* $\mathcal{N}^*$, *defined in* (2.21). *For these points we immediately get from* (2.36)

$$2\Phi_{m+\nu N}(z) = \left(\frac{\mathcal{T}_N(z) - \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)}{L} \right)^{\nu} \left(\Phi_m(z) - \frac{\mathcal{Q}_{m+l}(z)}{\sqrt{R(z)}} \right), \qquad z \in \mathcal{N}$$

$$2\Phi^*_{m+\nu N}(z) = \left(\frac{\mathcal{T}_N(z) - \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)}{L} \right)^{\nu} \left(\Phi^*_m(z) - \frac{\mathcal{Q}^*_{m+l}(z)}{\sqrt{R(z)}} \right), \qquad z \in \mathcal{N}^*. \tag{2.37}$$

Thus by (2.23) *the orthonormal polynomials and their reciprocal polynomials converge geometrically fast to zero on the finite sets* $\mathcal{N}$ *resp.* $\mathcal{N}^*$.

(b) *Let now* $\mathcal{K}$ *and* $\mathcal{K}^*$ *be arbitrary compact subsets of* $\mathbb{C} \backslash (\Gamma_{E_l} \cup \mathcal{N})$ *and of* $\mathbb{C} \backslash (\Gamma_{E_l} \cup \mathcal{N}^*)$, *respectively. Then by* (2.36) *and* (2.23) *there exists a positive value* $c > 0$ *and an index* n_0 *such that for all* $n \geq n_0$ *the following estimates hold*:

$$|\Phi_n(z)| \geq c \quad \textit{for all } z \in \mathcal{K} \qquad \textit{and} \qquad |\Phi^*_n(z)| \geq c \quad \textit{for all } z \in \mathcal{K}^*.$$

In the case of "periodic" orthogonal polynomials much more can be said on the location of the zeros than to lie in $|z| < 1$, which is known by the general theory (cf. [7, Theorem 9.1]). In order to be able to state our result we need the following notation.

Notation. Let $z \in \mathbb{C}$ and let $M \subset \mathbb{C}$. Then, as usual, we define

$$d(z, M) := \inf\{|z - y| : y \in M\}.$$

THEOREM 2.2. *Let $z_{j,n}$, $j=1, ..., n$, $n \in \mathbb{N}_0$, be the zeros of Φ_n. Then for every $\delta \in (0, 1)$ there is a $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ $\sup_{1 \leqslant j \leqslant n} d(z_{j,n}, \Gamma_{E_l} \cup \mathcal{N}) < \delta$. Moreover this implies that all accumulation points of the set $\bigcup_{n \in \mathbb{N}} \{z_{1,n}, ..., z_{n,n}\}$ belong to $\Gamma_{E_l} \cup \mathcal{N}$.*

Proof. The assertion immediately follows from Remark 2.1(b). ▮

Let us now give some estimates of the "periodic" orthonormal polynomials on the arcs Γ_{E_l}.

COROLLARY 2.2. *Let $\mathcal{E}$ be a closed subset of* Int(E_l). *Then there exist constants $k(\mathcal{E})$ and $K(\mathcal{E})$, which are independent of n, such that for $n \geqslant n_0$*

$$0 < k(\mathcal{E}) \leqslant |\Phi_n(e^{i\varphi})| \leqslant K(\mathcal{E}) < \infty \qquad \text{on } \mathcal{E}. \tag{2.38}$$

Further there exist constants α and β, again independent of n, such that for every $n \in \mathbb{N}_0$,

$$0 < \frac{1}{\alpha + n\beta} \leqslant |\Phi_n(e^{i\varphi})| \leqslant \alpha + n\beta < \infty \qquad \text{on } E_l. \tag{2.39}$$

There hold analog estimates as in (2.38) *and* (2.39) *for the polynomials of the second kind Ψ_n.*

Proof. First let us recall that the Ψ_n's, defined as in (2.22), can be considered to be orthonormal polynomials generated by the periodic reflection coefficients $\{-a_n\}_{n \in \mathbb{N}_0}$ and all of our already proven results about the polynomials Φ_n hold true for the Ψ_n's. To see this, one only has to exchange the parts of the P's and Ω's in the representations (2.12)–(2.15). Note that especially the polynomials R, $\mathcal{T}_N$, $\mathcal{U}_{N-l}$, and hence the set E_l remain, because of their symmetrical representation, the same.

By (1.6) and (2.22) there holds for all $\varphi \in [0, 2\pi]$

$$\tfrac{1}{2} e^{-in\varphi}(\Phi_n^*(e^{i\varphi})\, \Psi_n(e^{i\varphi}) + \Phi_n(e^{i\varphi})\, \Psi_n^*(e^{i\varphi})) = \text{Re}\{\Phi_n(e^{i\varphi})\, \overline{\Psi_n(e^{i\varphi})}\} = 1$$

and thus

$$|\Phi_n(e^{i\varphi})| \cdot |\Psi_n(e^{i\varphi})| \geqslant 1. \tag{2.40}$$

From the representations in Theorem 2.1(b) one obtains the uniform boundedness of the Φ_n's and Ψ_n's on $\mathcal{E}$ and together with (2.40) the assertion (2.38) follows.

In order to see the estimates in (2.39) note that in the same way as in (2.26) we get from (2.18) that

$$\sqrt{R}\,\Phi^*_{m+\nu N} \pm \mathcal{Q}^*_{(m+\nu N)+l} = \frac{\mathcal{T}_{\nu N} \pm \sqrt{R}\,\mathcal{U}_{\nu N-l}}{L_{\nu N}} (\sqrt{R}\,\Phi^*_m \pm \mathcal{Q}^*_{m+l})$$

for all $m \in \{0, ..., N-1\}$ and $\nu \in \mathbb{N}$, where $L_{\nu N} = 2\sqrt{d_{\nu N}} = L^\nu/2^{\nu-1}$. Thus we have for $\varphi \in E_l$, recall (2.26),

$$e^{i\frac{N\nu}{2}\varphi} e^{\pm i\nu\gamma} = \left(\frac{\mathcal{T}_N(e^{i\varphi}) \pm \sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})}{L} \right)^\nu$$

$$= 2^{\nu-1} \frac{\mathcal{T}_{\nu N}(e^{i\varphi}) \pm \sqrt{R(e^{i\varphi})}\,\mathcal{U}_{\nu N-l}(e^{i\varphi})}{L^\nu}.$$

Hence we can write for $\varphi \in E_l$,

$$\left| \frac{i \sin \nu\gamma}{\sqrt{R(e^{i\varphi})}} \right| = \frac{1}{2} \left| \frac{e^{i\nu\gamma} - e^{-i\nu\gamma}}{\sqrt{R(e^{i\varphi})}} \right| = \frac{2^{\nu-1}}{L^\nu} |\mathcal{U}_{\nu N-l}(e^{i\varphi})| \leqslant \left| \frac{\mathcal{U}_{N-l}(e^{i\varphi})}{L} \right| \cdot \nu,$$

where for the last estimate we have used the representation (2.20) and the fact that $|U_{\nu-1}(x)| \leqslant \nu$ for $x \in [-1, +1]$. Now the inequalities in (2.39) follow from Theorem 2.1(b) and (2.40). ∎

At the end of this section we prove the remarks we stated after (2.21) concerning the sets $\mathcal{N}$ and $\mathcal{N}^*$.

Remark 2.2. (a)

$$\mathcal{N} \cap \{z \in \mathbb{C}: |z| > 1\} = \varnothing \qquad \textit{and} \qquad \mathcal{N}^* \cap \{z \in \mathbb{C}: |z| < 1\} = \varnothing.$$

(b)

$$(\mathcal{N} \cap \{z \in \mathbb{C}: |z| = 1\}) = (\mathcal{N}^* \cap \{z \in \mathbb{C}: |z| = 1\}) = \{e^{i\xi_1}, ..., e^{i\xi_p}\}$$

and further

$$\mathcal{N} \subseteq \{z \in \mathbb{C}: P_N^{(m)}(z) = \Omega_N^{(m)}(z),\, m \in \{0, ..., N-1\}\} \cup \{e^{i\xi_1}, ..., e^{i\xi_p}\}. \quad (2.41)$$

Proof. (a) By [7, p. 42] and [8, formula (1.12′)] we have for all $n \in \mathbb{N}_0$

$$|\Phi_n(z)| \geqslant \frac{\sqrt{|z|^2 - 1}}{|z|} \quad \text{for } |z| > 1 \qquad \text{and} \qquad |\Phi_n^*(z)| \geqslant \sqrt{1 - |z|^2} \quad \text{for } |z| < 1.$$

(2.42)

Since the Φ_n tend to zero on $\mathcal{N}$ and the Φ_n^* tend to zero on $\mathcal{N}^*$, by Remark 2.1(a) it follows from (2.42) that $\mathcal{N}$ is a subset of the closed unit disk and $\mathcal{N}^*$ contains no points of $|z|<1$ (where the last fact could have been obtained also from (2.27) in the proof of Lemma 2.1).

(b) Let us first recall that $\mathcal{N} \cap \{z \in \mathbb{C}: |z|=1\} = \mathcal{N}^* \cap \{z \in \mathbb{C}: |z|=1\}$, as we mentioned in the lines after (2.21). Let now $z_0 \in \mathcal{N}$ with $|z_0|=1$; then we show that $z_0 = e^{i\xi_j}$ for a $j \in \{1, ..., p\}$, where the $e^{i\xi_j}$'s are those simple zeros of A at which a mass-point of $d\sigma(\varphi; A, W)$ occurs, i.e., $\xi_j \in \operatorname{supp}(\sigma(\varphi; A, W))$ and $\xi_j \notin E_l$; see (2.4). Indeed, suppose that $z_0 = e^{i\varphi_0}$ and $\varphi_0 \notin \operatorname{supp}(\sigma(\varphi; A, W))$. Then by [16, Lemma 3.1(d)], $\Phi_n(z_0) \to \infty$ as $n \to \infty$, which contradicts (2.37).

On the other hand we know from [17, Theorem 2.2 resp. Theorem 3.9] that at those points $\xi_1, ..., \xi_p$ where point-masses of $d\sigma(\varphi; A, W)$ appear there holds

$$\sqrt{R(e^{i\xi_j})}\, \Phi_n(e^{i\xi_j}) + \mathcal{Q}_{m+l}(e^{i\xi_j}) = 0 \qquad \text{for} \quad j = 1, ..., p,$$

i.e., $e^{i\xi_j} \in \mathcal{N}$.

To see (2.41) note that by (2.18) we have

$$RP^2_{m+N} - Q^2_{(m+N)+l} = \tfrac{1}{4} L^2 z^N (RP^2_m - Q^2_{m+l}), \qquad m \in \{0, ..., N-1\},$$

and hence from [17, Theorem 2.2] and [18, Corollary 4.2] it follows that

$$(\sqrt{R}\, P_m - Q_{m+l})(\sqrt{R} P_m + Q_{m+l}) = \frac{2 d_m z^m}{\mathcal{U}_{N-l}}\, VA(P_N^{(m)} - \Omega_N^{(m)}).$$

Since $(\sqrt{R}\, P_m - Q_{m+l})/z^m$ vanishes at all zeros $z_{p+1}, ..., z_{m^*}$ of A according to their multiplicity by (2.17) and since V and $\mathcal{U}_{N-l}$ have all their zeros on Γ_{E_l} (compare [18, Theorem 3.1(c)]), the assertion follows. ∎

3. ASYMPTOTIC PROPERTIES OF ORTHOGONAL POLYNOMIALS WITH ASYMPTOTICALLY PERIODIC REFLECTION COEFFICIENTS

Throughout this section let the sequence of reflection coefficients $\{a_n\}_{n \in \mathbb{N}_0}$ be such that

$$a_{n+N} = a_n \quad \text{for } n \in \mathbb{N}_0 \qquad \text{and} \qquad |a_j| < 1 \quad \text{for } j = 0, ..., N-1,\ N \in \mathbb{N} \text{ fixed},$$

that is, $\{a_n\}_{n\in\mathbb{N}_0}$ is purely periodic, and let $\{b_n\}_{n\in\mathbb{N}_0}$ be a sequence which is asymptotically periodic, i.e., which satisfies

$$\lim_{\nu\to\infty} b_{j+\nu N} = a_j \quad \text{for } j=0, ..., N-1 \qquad \text{and} \qquad |b_n|<1 \quad \text{for } n\in\mathbb{N}_0. \tag{3.1}$$

By $\{\tilde{P}_n\}_{n\in\mathbb{N}_0}$ we denote the sequence of orthogonal polynomials generated by the recurrence relation

$$\tilde{P}_{n+1}(z) = z\tilde{P}_n(z) - \bar{b}_n \tilde{P}_n^*(z), \qquad n\in\mathbb{N}_0, \quad \tilde{P}_0(z)=1 \tag{3.2}$$

and $\tilde{\Omega}_n$ denotes the monic polynomial of the second kind of $\tilde{P}_n$. As before, $\{P_n\}_{n\in\mathbb{N}_0}$ denotes the sequence of polynomials generated by the periodic reflection coefficients $\{a_n\}_{n\in\mathbb{N}_0}$ by (1.1).

In what follows we study the asymptotic behaviour of the polynomials $\tilde{\Phi}_n$, normed by

$$\tilde{\Phi}_n(z) := \frac{\tilde{P}_n(z)}{\sqrt{d_n}} \qquad \text{and} \qquad \tilde{\Psi}_n(z) := \frac{\tilde{\Omega}_n(z)}{\sqrt{d_n}}, \tag{3.3}$$

where the d_n's are given as in (1.6). Notice that in general the $\tilde{\Phi}_n$'s are not orthonormal polynomials, but the polynomials $\tilde{\Phi}_n$ and Φ_n, defined as in (2.22), have the same leading coefficient.

As already mentioned, we use the polynomials Φ_n, $n\in\mathbb{N}_0$, whose asymptotic behaviour we know from our results of Section 2, as "comparison system" to study asymptotic properties of the $\tilde{\Phi}_n$'s.

3.1. *Asymptotic Properties on* $\mathbb{C}\backslash\Gamma_{E_l}$

Some of the statements of this section will be proved by applying results we derived in [16], where we studied comparative asymptotics of orthogonal polynomials with perturbed reflection coefficients and obtained results on $|z|<1$ and under certain additional assumptions also on arcs of the unit circumference $|z|=1$. Because of the "periodicity" of the comparison system we now will be able to expand these results on the whole complex plane.

In order to be able to apply our results from [16] let us mention that from Theorem 2.1(a) and (2.23) one can derive that

$$\frac{1}{|\Phi_n^*(z)|^2} \sum_{\nu=0}^{n} |\Phi_\nu^*(z)|^2 \qquad \text{is uniformly bounded on } \mathcal{K}^*, \tag{3.4}$$

where $\mathcal{K}^*$ is a compact subset of $\mathbb{C}\backslash(\Gamma_{E_l}\cup\mathcal{N}^*)$, i.e., there exists an integer $n_0 = n_0(\mathcal{K}^*)$ and a constant $c = c(\mathcal{K}^*)$ such that the above expression is less than $c(\mathcal{K}^*)$ for all $n \geqslant n_0(\mathcal{K}^*)$ and for all $z\in\mathcal{K}^*$.

Henceforth, we will make essential use of the so-called nth functions of the second kind (introduced and investigated in detail by the first author in [14] and then by the authors in [16]), which are defined as

$$\mathcal{G}_n(z; A, W) := \frac{1}{z^n} (\Phi_n(z) F(z; A, W) + \Psi_n(z))$$

$$\mathcal{H}_n(z; A, W) := \frac{1}{z^{n+1}} (\Phi_n^*(z) F(z; A, W) - \Psi_n^*(z))$$

$$\text{for} \quad z \in \mathbb{C} \backslash \{e^{i\xi_1}, ..., e^{i\xi_p}\} \qquad \text{and} \qquad V(z) \neq 0 \tag{3.5}$$

where $F(e^{i\varphi}; A, W) := \lim_{r \to 1^-} F(re^{i\varphi}; A, W)$ for $\varphi \in E_l$. Here we have used the same notation as in Section 2.

Note that the functions $\mathcal{G}_n$ and $\mathcal{H}_n$ have simple poles at $e^{i\xi_1}, ..., e^{i\xi_p}$ and poles of order $\frac{1}{2}$ at those boundary-points $e^{i\varphi_j}$ of Γ_{E_l} where $V(e^{i\varphi_j}) = 0$. By these definitions we immediately get that

$$\mathcal{G}_n(z; A, W) = -\frac{1}{z^{n+1}} \overline{\mathcal{H}_n(1/\bar{z}; A, W)}, \qquad z \in \mathbb{C} \backslash (\Gamma_{E_l} \cup \{e^{i\xi_1}, ..., e^{i\xi_p}\}). \tag{3.6}$$

If one considers (2.8), (2.16), (2.26), and (2.33), the following explicit representations can be obtained, $m \in \mathbb{N}_0$ and $v \in \mathbb{N}$,

$$\mathcal{G}_{m+vN}(z; A, W) = \frac{1}{z^{m+vN} V(z) A(z)} \left(\frac{\mathcal{T}_N(z) - \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)}{L} \right)^v \times (\sqrt{R(z)}\, \Phi_m(z) - \mathcal{Q}_{m+l}(z)) \tag{3.7}$$

$$\mathcal{H}_{m+vN}(z; A, W) = \frac{1}{z^{m+vN+1} V(z) A(z)} \left(\frac{\mathcal{T}_N(z) - \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)}{L} \right)^v \times (\sqrt{R(z)}\, \Phi_m^*(z) - \mathcal{Q}_{m+l}^*(z)).$$

Hence, $\mathcal{G}_n$ and $\mathcal{H}_n$ converge to zero uniformly compact and geometrically fast on $\mathbb{C} \backslash (\Gamma_{E_l} \cup \{e^{i\xi_1}, ..., e^{i\xi_p}\})$ by (2.17) and (2.23) resp. (2.30). Further we get from [16, (1.34) and (1.35)] and (3.6) that for all $n \in \mathbb{N}_0$,

$$|\mathcal{H}_n(z; A, W)| \leqslant |\mathcal{G}_n(z; A, W)| \qquad \text{for} \quad |z| < 1$$

$$|\mathcal{H}_n(z; A, W)| = |\mathcal{G}_n(z; A, W)| \qquad \text{for} \quad z = e^{i\varphi} \text{ and } \varphi \notin E_l \tag{3.8}$$

$$|\mathcal{H}_n(z; A, W)| \geqslant |\mathcal{G}_n(z; A, W)| \qquad \text{for} \quad |z| > 1.$$

For some of our proofs we will also need the following well-known facts (cf. e.g. [8, (1.7)])

$$\begin{aligned} |\Phi_n(z)| &\leqslant |\Phi_n^*(z)| && \text{for} \quad |z|<1 \\ |\Phi_n(z)| &= |\Phi_n^*(z)| && \text{for} \quad z=e^{i\varphi} \text{ and } \varphi\in[0,2\pi] \\ |\Phi_n(z)| &\geqslant |\Phi_n^*(z)| && \text{for} \quad |z|>1. \end{aligned} \tag{3.9}$$

Further, considering the representations in (3.7) and (2.36) one obtains by using (2.9), (2.17), (2.23), and (2.30) that the functions

$$\Phi_n^{[*]}(z)\,\mathscr{G}_n(z;A,W) \qquad \text{and} \qquad \Phi_n^{[*]}(z)\,\mathscr{H}_n(z;A,W) \tag{3.10}$$

are uniformly bounded on every compact subset of $\mathbb{C}\backslash(\Gamma_{E_l}\cup\{e^{i\xi_1},...,e^{i\xi_p}\})$, where again $\Phi_n^{[*]}$ means either Φ_n or Φ_n^*.

Finally let us mention that from (1.6) and (2.22) it follows that

$$\Phi_n^*(z)\,\mathscr{G}_n(z;A,W)-z\Phi_n(z)\,\mathscr{H}_n(z;A,W)=2, \qquad \text{for} \quad z\in\mathbb{C}\backslash\{e^{i\xi_1},...,e^{i\xi_p}\} \quad \text{and} \quad V(z)\neq 0. \tag{3.11}$$

The next theorem is essential in order to obtain comparative asymptotics.

THEOREM 3.1. *Let the polynomials Φ_n, $\tilde{\Phi}_n$ and the functions of the second kind $\mathscr{G}_n$, $\mathscr{H}_n$ be given as above. In addition to* (3.1) *we assume that*

$$\sum_{n=0}^{\infty}|a_n-b_n|<\infty. \tag{3.12}$$

Then there exists an analytic function Δ on $\mathbb{C}\backslash(\Gamma_{E_l}\cup\{e^{i\xi_1},...,e^{i\xi_p}\})$ such that

$$\lim_{n\to\infty}(\tilde{\Phi}_n^*(z)\,\mathscr{G}_n(z;A,W)-z\tilde{\Phi}_n(z)\,\mathscr{H}_n(z;A,W))=\Delta(z) \tag{3.13}$$

uniformly on every closed (*not necessarily bounded*) *subset of $\mathbb{C}\backslash(\Gamma_{E_l}\cup\{e^{i\xi_1},...,e^{i\xi_p}\})$.*

Proof. Let $\mathscr{K}$ be an arbitrary compact subset of $\mathbb{C}\backslash(\Gamma_{E_l}\cup\{e^{i\xi_1},...,e^{i\xi_p}\})$. We will show that

$$\tilde{\Phi}_n^*(z)\,\mathscr{G}_n(z;A,W) \quad \text{and} \quad z\tilde{\Phi}_n(z)\,\mathscr{H}_n(z;A,W) \text{ are uniformly bounded on } \mathscr{K}. \tag{3.14}$$

By

$$|z\tilde{\Phi}_n(z)\,\mathscr{H}_n(z;A,W)| = \left|\tilde{\Phi}_n^*\left(\frac{1}{\bar{z}}\right)\mathscr{G}_n\left(\frac{1}{\bar{z}};A,W\right)\right|,$$

which follows from (3.6), and by (3.8), (3.9) it sufficies to show only the uniform boundedness of $\tilde{\Phi}_n^*\mathscr{G}_n(\,\cdot\,;A,W)$ on $\mathscr{K}_I:=\{z\in\mathbb{C}: z\in\mathscr{K}$ and $|z|\leqslant 1\}$. From [16, Lemma 2.1(c)] one can derive that

$$\left|\frac{\tilde{\Phi}_n^*(z)}{\Phi_n^*(z)}\right| \leqslant c_1 + c_2\sum_{v=0}^{n-1}|a_v-b_v|\left\{(|\Phi_{n-v}^{(v)*}(z)|+|\Psi_{n-v}^{(v)*}(z)|)\left|\frac{\Phi_v^*(z)}{\Phi_n^*(z)}\right|\right\}$$
$$\times\left|\frac{\tilde{\Phi}_v^*(z)}{\Phi_v^*(z)}\right| \qquad \text{uniformly on } \mathscr{K}_I$$

(note that the Φ_v^*'s are bounded away from zero on $\mathscr{K}_I$), where c_1 and c_2 are positive constants and where

$$\Phi_{n-v}^{(v)} := \frac{P_{n-v}^{(v)}}{\sqrt{d_{n-v}^{(v)}}}, \qquad \Psi_{n-v}^{(v)} := \frac{\Omega_{n-v}^{(v)}}{\sqrt{d_{n-v}^{(v)}}}, \qquad \text{with} \quad d_{n-v}^{(v)} := \prod_{j=v}^{n-1}(1-|a_j|^2).$$

Now recall that the reflection coefficients $\{a_{n+v}\}_{n\in\mathbb{N}_0}$ of the $P_n^{(v)}$'s are periodic and that by (2.12) and (2.15) the a_{n+v}'s generate the same polynomials R, $\mathscr{T}_N$, $\mathscr{U}_{N-l}$ and the same set E_l as the original reflection coefficients $\{a_n\}_{n\in\mathbb{N}_0}$. Hence there hold analog representations as in (2.36) for the $\Phi_{n-v}^{(v)}$'s and $\Psi_{n-v}^{(v)}$'s (for the $\Psi_{n-v}^{(v)}$'s compare the proof of Corollary 2.2). Since $\Phi_m^{(v)}=\Phi_m^{(v+N)}$ and $\Psi_m^{(v)}=\Psi_m^{(v+N)}$ the uniform boundedness on $\mathscr{K}_I$ of the $\{\cdots\}$-term in the above sum can be derived by Theorem 2.1(a) and (2.9). Hence, there exists a positive constant c_3 such that uniformly on $\mathscr{K}_I$,

$$\left|\frac{\tilde{\Phi}_n^*(z)}{\Phi_n^*(z)}\right| \leqslant c_1 + c_3\sum_{v=0}^{n-1}|a_v-b_v|\cdot\left|\frac{\tilde{\Phi}_v^*(z)}{\Phi_v^*(z)}\right|.$$

From (3.12) and Gronwall's inequality (see e.g. [16, (1.37)]) we obtain the uniform boundedness of the sequence $\{\tilde{\Phi}_n^*/\Phi_n^*\}$ on $\mathscr{K}_I$.

Now we can write

$$|\tilde{\Phi}_n^*(z)\,\mathscr{G}_n(z;A,W)| = \left|\frac{\tilde{\Phi}_n^*(z)}{\Phi_n^*(z)}\right|\cdot|\Phi_n^*(z)\,\mathscr{G}_n(z;A,W)|$$

and the uniform boundedness of the functions $\tilde{\Phi}_n^*\mathscr{G}_n(\,\cdot\,;A,W)$ on $\mathscr{K}_I$ follows from (3.10). Thus we have shown (3.14).

Because of (3.14) the functions

$$\Delta_n(z) := \tilde{\Phi}_n^*(z)\, \mathscr{G}_n(z; A, W) - z\tilde{\Phi}_n(z)\, \mathscr{H}_n(z; A, W), \qquad n \in \mathbb{N}_0,$$

are uniformly bounded on $\mathscr{K}$ and they are further analytic on $\mathbb{C}\backslash(\Gamma_{E_l} \cup \{e^{i\xi_1}, ..., e^{i\xi_p}\})$. From [16, Theorem 2.2], note (3.4), we know that

$$\lim_{n\to\infty} \Delta_n(z) =: \Delta(z) \qquad \text{uniformly on } |z| \leqslant r < 1.$$

Now we can apply Vitali's theorem on $\mathscr{K}$ and (3.13) follows on every compact subset of $\mathbb{C}\backslash(\Gamma_{E_l} \cup \{e^{i\xi_1}, ..., e^{i\xi_p}\})$. From (3.6) we see that

$$\Delta_n(z) = \overline{\Delta_n\left(\frac{1}{\bar{z}}\right)}, \tag{3.15}$$

hence (3.13) even holds on every closed, not necessarily bounded, subset of $\mathbb{C}\backslash(\Gamma_{E_l} \cup \{e^{i\xi_1}, ..., e^{i\xi_p}\})$. ∎

Let us point out that the periodicity of the reflection coefficients of the comparison system implies that the limit relation (3.13) holds even on $\mathbb{C}\backslash(\Gamma_{E_l} \cup \{e^{i\xi_1}, ..., e^{i\xi_p}\})$ and not only on $|z| \leqslant r < 1$ as in the general case (see [16, Theorem 2.2]).

The next theorem shows how the undisturbed "periodic" and the disturbed "asymptotically periodic" polynomials are asymptotically related to each other.

THEOREM 3.2. *Let the assumptions of Theorem* 3.1 *be fulfilled and let the function* $\Delta(z)$ *be given as in* (3.13). *Then*

$$\lim_{n\to\infty} \left(\frac{\tilde{\Phi}_n^*(z)}{\Phi_n^*(z)} - \frac{1}{2}\Delta(z)\right) = 0 \qquad \text{on} \quad \mathbb{C}\backslash(\Gamma_{E_l} \cup \mathcal{N}^*) \tag{3.16}$$

$$\lim_{n\to\infty} \left(\frac{\tilde{\Phi}_n(z)}{\Phi_n(z)} - \frac{1}{2}\Delta(z)\right) = 0 \qquad \text{on} \quad \mathbb{C}\backslash(\Gamma_{E_l} \cup \mathcal{N}), \tag{3.17}$$

where both convergences hold uniformly compact on $\mathbb{C}\backslash(\Gamma_{E_l} \cup \mathcal{N}^*)$ *resp.* $\mathbb{C}\backslash(\Gamma_{E_l} \cup \mathcal{N})$.

Proof. Let $\mathscr{K}^*$ be a compact subset of $\mathbb{C}\backslash(\Gamma_{E_l} \cup \mathcal{N}^*)$. Then by Remark 2.1(b) the Φ_n^*'s are uniformly bounded away from zero on $\mathscr{K}^*$ from a certain index n_0 onward. We will show that

$$\frac{\tilde{\Phi}_n^*(z)}{\Phi_n^*(z)} \quad \text{is uniformly bounded on } \mathscr{K}^* \quad (\text{for } n \geqslant n_0). \tag{3.18}$$

In the proof of Theorem 3.1 we have already shown that (3.18) holds on $\mathcal{K}^* \cap \{z \in \mathbb{C}: |z| \leqslant 1\}$ (note that by Remark 2.2 the $e^{i\xi_j}$, $j=1, ..., p$, are exactly that points in $\mathcal{N}^*$ with modulus 1). Let now $\mathcal{K}^*_o :=$ $\mathcal{K}^* \cap \{z \in \mathbb{C}: |z| \geqslant 1\}$; then by (3.9),

$$\begin{aligned} &|z\tilde{\Phi}_n(z)\, \mathcal{H}_n(z; A, W)| \\ &\quad \geqslant |z\tilde{\Phi}^*_n(z)\, \mathcal{H}_n(z; A, W)| \\ &\quad = \left|\frac{\tilde{\Phi}^*_n(z)}{\Phi^*_n(z)}\right| \left|\frac{\Phi^*_n(z)}{\Phi_n(z)}\right| |z\Phi_n(z)\, \mathcal{H}_n(z; A, W)| \qquad \text{on} \quad \mathcal{K}^*_o. \quad (3.19) \end{aligned}$$

Let us note that for all $n \in \mathbb{N}$ there holds

$$|z\Phi_n(z)\, \mathcal{H}_n(z; A, W)| \geqslant 1 \qquad \text{on} \quad \mathcal{K}^*_o. \tag{3.20}$$

Indeed, assume the opposite $|z_0 \Phi_{\nu_0}(z_0)\, \mathcal{H}_{\nu_0}(z_0; A, W)| < 1$ for a fixed $\nu_0 \in \mathbb{N}$ and $z_0 \in \mathcal{K}^*_o$, then we have by (3.8) and (3.9)

$$\begin{aligned} &|\Phi^*_{\nu_0}(z_0)\, \mathcal{G}_{\nu_0}(z_0; A, W) - z_0 \Phi_{\nu_0}(z_0)\, \mathcal{H}_{\nu_0}(z_0; A, W)| \\ &\qquad \leqslant 2\, |z_0 \Phi_{\nu_0}(z_0)\, \mathcal{H}_{\nu_0}(z_0; A, W)| < 2, \end{aligned}$$

which contradicts (3.11). Now the uniform boundedness of the sequence $\{\tilde{\Phi}^*_n/\Phi^*_n\}$ on $\mathcal{K}^*_o$ follows from (3.19), (3.14) and (3.20), where one has to consider the fact that by Theorem 2.1(a) the Φ^*_n/Φ_n's are uniformly bounded away from zero on $\mathcal{K}^*_o$ from a certain index onward. Thus we have shown (3.18).

By (3.4) and [16, Theorem 2.2] the limit-relation (3.16) holds uniformly on $|z| \leqslant r < 1$ and by (3.18) and Vitali's theorem the uniform convergence on compact subsets of $\mathbb{C} \backslash (\Gamma_{E_l} \cup \mathcal{N}^*)$ follows.

In order to see (3.17) note that for every $z \in \mathbb{C} \backslash (\Gamma_{E_l} \cup \mathcal{N} \cup \{0\})$ we have by (3.15) and the definition of the reciprocal polynomials

$$\frac{\tilde{\Phi}_n(z)}{\Phi_n(z)} - \frac{1}{2}\Delta(z) = \overline{\frac{\tilde{\Phi}^*_n\left(\frac{1}{\bar{z}}\right)}{\Phi^*_n\left(\frac{1}{\bar{z}}\right)} - \frac{1}{2}\Delta\left(\frac{1}{\bar{z}}\right)}. \tag{3.21}$$

Now $1/\bar{z} \in \mathbb{C} \backslash (\Gamma_{E_l} \cup \mathcal{N}^*)$ and the assertion follows from (3.16) on every compact subset of $\mathbb{C} \backslash (\Gamma_{E_l} \cup \mathcal{N} \cup \{0\})$. If $0 \notin \mathcal{N}$ then we can write

$$|\tilde{\Phi}_n(z)\, \mathcal{G}_n(z; A, W)| = \left|\frac{\tilde{\Phi}_n(z)}{\Phi_n(z)}\right| \left|\frac{\Phi_n(z)}{\Phi^*_n(z)}\right| |\Phi^*_n(z)\, \mathcal{G}_n(z; A, W)|,$$

which implies the uniform boundedness of the sequence $\{\tilde{\Phi}_n/\Phi_n\}$ in a neighbourhood of $z=0$ (compare the proof of (3.18) above). In this case the convergence on $\mathbb{C}\setminus(\Gamma_{E_l}\cup\mathcal{N}\cup\{0\})$ and Vitali's theorem give the uniform convergence on compact subsets of $\mathbb{C}\setminus(\Gamma_{E_l}\cup\mathcal{N})$. ▌

REMARK 3.1. *From* (3.21) *we see that the convergence in* (3.17) *even holds on closed, not necessarily bounded, subsets of* $\mathbb{C}\setminus(\Gamma_{E_l}\cup\mathcal{N})$. *The same holds true for* (3.16) *and every closed subset of* $\mathbb{C}\setminus(\Gamma_{E_l}\cup\mathcal{N}^*)$ *if* $0\notin\mathcal{N}$.

Let us now define the set Z_Δ by the zeros of the function Δ, i.e.,

$$Z_\Delta:=\{z\in\mathbb{C}\colon \Delta(z)=0\}. \tag{3.22}$$

Then Z_Δ has the following fundamental properties.

REMARK 3.2. *In* [16, *Lemma* 2.2] *we have shown that the function* Δ *has no zeros on* $|z|<1$ *and thus by* (3.15) Δ *cannot vanish on* $|z|>1$, *too. Hence,* $Z_\Delta\subset\{z\in\mathbb{C}\colon |z|=1\}$ *and, since* Δ *is analytic and not identically zero on* $\mathbb{C}\setminus(\Gamma_{E_l}\cup\{e^{i\xi_1},...,e^{i\xi_p}\})$,

$$Z_\Delta\setminus\{e^{i\varphi}\colon \varphi\in\mathcal{M}\}\qquad \textit{is finite}$$

for every compact subset $\mathcal{M}$ *of* $[0,2\pi]$ *with* $E_l\cup\{\xi_1,...,\xi_p\}\subset\operatorname{Int}(\mathcal{M})$. *Thus* Z_Δ *is at most a countable set. Further we claim that*

$$\{e^{i\varphi}\colon \varphi\in\operatorname{supp}(\tilde{\sigma})\setminus E_l\}\subseteq Z_\Delta\cup\{e^{i\xi_1},...,e^{i\xi_p}\},$$

where $\tilde{\sigma}$ *denotes the orthogonality measure of the* $\tilde{\Phi}_n$*'s. Indeed, by using the same methods as in the proof of* [9, *Theorem* 3] *we get that* $\operatorname{supp}(\tilde{\sigma})\setminus E_l$ *consists of at most countable-many mass-points and thus by* [10, (7) *and* (11)] $\sum_{n=0}^{\infty}|\tilde{\Phi}_n^*(e^{i\psi})|^2<\infty$ *for all* $\psi\in\operatorname{supp}(\tilde{\sigma})\setminus E_l$. *This last convergence is only possible if* $\Delta(e^{i\psi})=0$ *or* $\psi\in\{\xi_1,...,\xi_p\}$, *because the other case, i.e.,* $|\Delta(e^{i\psi})|>0$ *and* $\psi\notin\{\xi_1,...,\xi_p\}$, *yields by Theorem* 3.2 *that*

$$|\tilde{\Phi}_n^*(e^{i\psi})|\geqslant\tfrac{1}{4}|\Delta(e^{i\psi})|\cdot|\Phi_n^*(e^{i\psi})|$$

for all $n\geqslant n_0$, *which means that* $\tilde{\Phi}_n^*(e^{i\psi})\xrightarrow[n\to\infty]{}\infty$ *by Theorem* 2.1 *and Remark* 2.2.

The following theorem says that the zeros of the perturbed orthogonal polynomials $\tilde{\Phi}_n$ behave in the same way as the zeros of the undisturbed orthogonal polynomials Φ_n, described in Theorem 2.2.

THEOREM 3.3. *Let the assumptions of Theorem* 3.1 *be fulfilled. Let* $\tilde{z}_{j,n}$, $j=1,...,n$, $n\in\mathbb{N}_0$, *be the zeros of* $\tilde{\Phi}_n$. *Then for every* $\delta\in(0,1)$ *there is a*

$n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ $\sup_{1 \leqslant j \leqslant n} d(\tilde{z}_{j,n}, \Gamma_{E_l} \cup \mathcal{N} \cup Z_\Delta) < \delta$, where $Z_\Delta \subseteq \{z \in \mathbb{C}: |z| = 1\}$ and Z_Δ is at most countable. Moreover this implies that all accumulation points of the set $\bigcup_{n \in \mathbb{N}} \{\tilde{z}_{1,n}, ..., \tilde{z}_{n,n}\}$ belong to $\Gamma_{E_l} \cup \mathcal{N} \cup Z_\Delta$.

Proof. From Theorem 2.1(a) and Theorem 3.2 we get the asymptotical behaviour of the perturbed polynomials $\tilde{\Phi}_n$. Now the assertions follow from Theorem 2.2 and Remark 3.2. ▮

Let us mention that the location resp. the asymptotically distribution of the zeros of orthogonal polynomials whose reflection coefficients $\{b_n\}_{n \in \mathbb{N}_0}$ are in the Szegö-class, i.e., satisfy the condition $\sum_{n=0}^{\infty} |b_n|^2 < \infty$ and thus are not of the type considered here, has been studied in [6, 11, 12].

3.2. *Asymptotic Properties on Γ_{E_l}*

Let us recall that by our notation (2.34), $\sqrt{R(e^{i\varphi})}$ denotes one of the boundary values $\sqrt[-]{R(e^{i\varphi})}$ or $\sqrt[+]{R(e^{i\varphi})}$.

By Corollary 2.2 the polynomials Φ_n and Ψ_n are uniformly bounded on every compact subset $\mathscr{E}$ of $\operatorname{Int}(E_l)$. The following lemma gives the uniform boundedness of $\tilde{\Phi}_n$ and $\tilde{\Psi}_n$ on $\mathscr{E}$ and is important for the following proofs.

LEMMA 3.1. *Let the assumption* (3.12) *be fulfilled. Then the polynomials $\tilde{\Phi}_n$ and $\tilde{\Psi}_n$ are uniformly bounded away from zero and from infinity on every compact subset $\mathscr{E}$ of* $\operatorname{Int}(E_l)$.

Proof. The uniform boundedness from infinity follows from [16, Proposition 3.1] and Corollary 2.2, and the uniform boundedness from zero can be obtained from an analog property as given in (2.40), which holds as well for the polynomials $\tilde{\Phi}_n$ and $\tilde{\Psi}_n$. ▮

Motivated by (2.9)–(2.15) let us define, $n \in \mathbb{N}_0$,

$$\mathfrak{R}^{[n]} := \frac{1}{4}\left[(\tilde{P}_N^{(n)} + \tilde{\Omega}_N^{(n)} + \tilde{P}_N^{(n)*} + \tilde{\Omega}_N^{(n)*})^2 - 16 z^N \prod_{j=0}^{N-1} (1 - |b_{n+j}|^2)\right] \tag{3.23}$$

$$\mathscr{T}_N^{[n]} := \frac{1}{2}(\tilde{P}_N^{(n)} + \tilde{\Omega}_N^{(n)} + \tilde{P}_N^{(n)*} + \tilde{\Omega}_N^{(n)*}) \tag{3.24}$$

$$\mathfrak{U}^{[n]} := \frac{1}{d_n}[\tilde{P}_n \tilde{P}_{n+N}^* - \tilde{P}_n^* \tilde{P}_{n+N}] = \sqrt{d_N}\,[\tilde{\Phi}_n \tilde{\Phi}_{n+N}^* - \tilde{\Phi}_n^* \tilde{\Phi}_{n+N}] \tag{3.25}$$

and

$$\tilde{L}^{[n]} := \sqrt{4 \prod_{j=0}^{N-1} (1 - |b_{n+j}|^2)}. \tag{3.26}$$

Then, as in (2.9), there holds a relation of the form

$$(\tilde{\mathcal{T}}_N^{[n]})^2(z) - \tilde{\mathfrak{R}}^{[n]}(z) = (\tilde{L}^{[n]})^2 z^N. \tag{3.27}$$

From (3.1), (2.9), (2.12), and (2.15), one immediately obtains

$$\tilde{\mathfrak{R}}^{[n]}(z) \xrightarrow[n\to\infty]{} R(z)\,\mathcal{U}_{N-l}^2(z), \qquad \tilde{\mathcal{T}}_N^{[n]}(z) \xrightarrow[n\to\infty]{} \mathcal{T}_N(z),$$
$$\tilde{L}^{[n]} \xrightarrow[n\to\infty]{} L \quad \text{uniformly compact.} \tag{3.28}$$

Next let us introduce a new sequence of monic orthogonal polynomials $\{\tilde{P}_\nu^{[n]}\}_{\nu\in\mathbb{N}_0}$ generated by the reflection coefficients $\{\tilde{b}_\nu^{[n]}\}_{\nu\in\mathbb{N}_0}$, which are obtained by repeating periodically the N reflection coefficients $b_n, ..., b_{n+N-1}$ of $\{\tilde{P}_\nu\}_{\nu\in\mathbb{N}_0}$ from the index n onward, i.e.,

$$\tilde{b}_j^{[n]} := b_j \quad \text{for } j = 0, ..., n+N-1 \qquad \text{and} \qquad \tilde{b}_j^{[n]} := \tilde{b}_{j-N}^{[n]} \quad \text{for } j \geqslant n+N.$$

Note that by this definition

$$\tilde{P}_\nu = \tilde{P}_\nu^{[n]} \qquad \text{for} \quad \nu = 0, ..., n+N. \tag{3.29}$$

Finally let us define the polynomials $\tilde{Q}_{\nu+N}^{[n]}$, compare (2.16) resp. (2.18),

$$\tilde{Q}_{\nu+N}^{[n]}(z) := 2\tilde{P}_{\nu+N}^{[n]}(z) - \tilde{\mathcal{T}}_N^{[n]}(z)\,\tilde{P}_\nu^{[n]}(z) = z^{\nu+N} + \cdots, \quad \nu, n \in \mathbb{N}_0. \tag{3.30}$$

In order to get explicit representations of the perturbed polynomials $\tilde{\Phi}_n$ on Γ_{E_l} we need some preliminary considerations. Let us first state that from [18, Corollary 4.1(a)] it follows that the undisturbed "periodic" polynomials satisfy

$$P_{m+(\nu+2)N} - \tfrac{1}{2}(\mathcal{T}_N \pm \sqrt{R}\,\mathcal{U}_{N-l})\,P_{m+(\nu+1)N}$$
$$= \left[\frac{\mathcal{T}_N \mp \sqrt{R}\,\mathcal{U}_{N-l}}{2}\right]^{\nu+1} (P_{m+N} - \tfrac{1}{2}(\mathcal{T}_N \pm \sqrt{R}\,\mathcal{U}_{N-l})\,P_m), \quad m, \nu \in \mathbb{N}_0.$$

For the polynomials $\{\tilde{P}_n\}_{n\in\mathbb{N}_0}$ there holds

LEMMA 3.2. *Let $m, \nu \in \mathbb{N}_0$. Then we have*

$$\tilde{P}_{m+(\nu+2)N} - \tfrac{1}{2}(\mathcal{T}_N \pm \sqrt{R}\,\mathcal{U}_{N-l})\,\tilde{P}_{m+(\nu+1)N}$$
$$= \left[\frac{\mathcal{T}_N \mp \sqrt{R}\,\mathcal{U}_{N-l}}{2}\right]^{\nu+1} (\tilde{P}_{m+N} - \tfrac{1}{2}(\mathcal{T}_N \pm \sqrt{R}\,\mathcal{U}_{N-l})\,\tilde{P}_m)$$
$$+ \sum_{j=0}^{\nu} \left[\frac{\mathcal{T}_N \mp \sqrt{R}\,\mathcal{U}_{N-l}}{2}\right]^{\nu-j} \delta_{m+jN},$$

where

$$\delta_n(z) := (e_n(z) + \tilde{\mathcal{T}}_N^{[n]}(z) - \mathcal{T}_N(z))\, \tilde{P}_{n+N}(z) + f_n(z)\, \tilde{P}^*_{n+N}(z)$$
$$+ \frac{z^N}{4} (L^2 - (\tilde{L}^{[n]})^2)\, \tilde{P}_n(z)$$

$$e_n(z) := \frac{1}{2} (\tilde{P}_N^{(n+N)}(z) + \tilde{\Omega}_N^{(n+N)}(z) - \tilde{P}_N^{(n)}(z) - \tilde{\Omega}_N^{(n)}(z))$$

$$f_n(z) := \frac{1}{2} (\tilde{P}_N^{(n+N)}(z) - \tilde{\Omega}_N^{(n+N)}(z) - \tilde{P}_N^{(n)}(z) + \tilde{\Omega}_N^{(n)}(z)).$$

Proof. By the above definitions (3.23)–(3.30) we can write

$$\tilde{Q}^{[m]}_{m+N}(z) = 2\tilde{P}_{m+N}(z) - \tilde{\mathcal{T}}_N^{[m]}(z)\, \tilde{P}_m(z)$$
$$\tilde{Q}^{[m]}_{m+2N}(z) = 2\tilde{P}^{[m]}_{m+2N}(z) - \tilde{\mathcal{T}}_N^{[m]}(z)\, \tilde{P}_{m+N}(z).$$

Thus we have the representations

$$2\tilde{P}_{m+N} = \tilde{\mathcal{T}}_N^{[m]}\, \tilde{P}_m + \tilde{Q}^{[m]}_{m+N}$$
$$2\tilde{P}_{m+2N} = \tilde{\mathcal{T}}_N^{[m+N]}\, \tilde{P}_{m+N} + \tilde{Q}^{[m+N]}_{m+2N}$$
$$2\tilde{Q}^{[m]}_{m+2N} = \tilde{\mathcal{T}}_N^{[m]}\, \tilde{Q}^{[m]}_{m+N} + \tilde{\mathfrak{R}}^{[m]} \tilde{P}_m$$

and we obtain

$$2\tilde{P}_{m+2N} = \tilde{\mathcal{T}}_N^{[m+N]}\, \tilde{P}_{m+N} + \tilde{Q}^{[m]}_{m+2N} + (\tilde{Q}^{[m+N]}_{m+2N} - \tilde{Q}^{[m]}_{m+2N})$$
$$= 2\mathcal{T}_N \tilde{P}_{m+N} - \frac{L^2 z^N}{2}\, \tilde{P}_m$$
$$+ \Bigg[(\tilde{Q}^{[m+N]}_{m+2N} - \tilde{Q}^{[m]}_{m+2N}) + (\tilde{\mathcal{T}}_N^{[m+N]} + \tilde{\mathcal{T}}_N^{[m]} - 2\mathcal{T}_N)\, \tilde{P}_{m+N}$$
$$+ (L^2 - (\tilde{L}^{[m]})^2)\, \frac{z^N}{2}\, \tilde{P}_m \Bigg].$$

By using the two identities

$$\tilde{Q}^{[m+N]}_{m+2N} - \tilde{Q}^{[m]}_{m+2N} = 2(\tilde{P}_{m+2N} - \tilde{P}^{[m]}_{m+2N}) - (\tilde{\mathcal{T}}_N^{[m+N]} - \tilde{\mathcal{T}}_N^{[m]})\, \tilde{P}_{m+N}$$

and (see [18, (1.15) and (1.16)])

$$\begin{aligned}\tilde{P}_{m+2N} - \tilde{P}^{[m]}_{m+2N} &= \tfrac{1}{2}[\tilde{P}^{(m+N)}_N + \tilde{\Omega}^{(m+N)}_N - \tilde{P}^{(m)}_N - \tilde{\Omega}^{(m)}_N]\, \tilde{P}_{m+N} \\ &\quad + \tfrac{1}{2}[\tilde{P}^{(m+N)}_N - \tilde{\Omega}^{(m+N)}_N - \tilde{P}^{(m)}_N + \tilde{\Omega}^{(m)}_N]\, \tilde{P}^*_{m+N} \\ &= e_m \tilde{P}_{m+N} + f_m \tilde{P}^*_{m+N}\end{aligned}$$

we can write

$$\begin{aligned}\tilde{P}_{m+2N} &= \mathcal{T}_N \tilde{P}_{m+N} - \frac{L^2 z^N}{4}\tilde{P}_m + \Big[(e_m + \tilde{\mathcal{T}}^{[m]}_N - \mathcal{T}_N)\, \tilde{P}_{m+N} \\ &\quad + f_m \tilde{P}^*_{m+N} + \frac{z^N}{4}(L^2 - (\tilde{L}^{[m]})^2)\, \tilde{P}_m \Big] \\ &= \mathcal{T}_N \tilde{P}_{m+N} - \frac{1}{4}(\mathcal{T}^2_N - R\,\mathcal{U}^2_{N-l})\, \tilde{P}_m + \delta_m,\end{aligned}$$

i.e.,

$$\begin{aligned}&\tilde{P}_{m+2N} - \tfrac{1}{2}(\mathcal{T}_N \pm \sqrt{R}\,\mathcal{U}_{N-l})\, \tilde{P}_{m+N} \\ &\quad = \tfrac{1}{2}(\mathcal{T}_N \mp \sqrt{R}\,\mathcal{U}_{N-l})(\tilde{P}_{m+N} - \tfrac{1}{2}(\mathcal{T}_N \pm \sqrt{R}\,\mathcal{U}_{N-l})\, \tilde{P}_m) + \delta_m.\end{aligned}$$

From the last equation the assertion follows by iteration. ∎

Let us now define

$$\Theta_{m+jN}(z) := \left(\prod_{\nu=0}^{m+jN-1} (1 - |a_\nu|^2) \right)^{-\frac{1}{2}} \delta_{m+jN}(z), \qquad m, j \in \mathbb{N}_0, \tag{3.31}$$

where δ_{m+jN} is given as in Lemma 3.2. Then there holds

LEMMA 3.3. *Let $\mathcal{E}$ be a closed subset of* $\operatorname{Int}(E_l)$ *and let assumption* (3.12) *be fulfilled. Then*

$$|\Theta_n(e^{i\varphi})| \leqslant const \cdot \sum_{\nu=n}^{n+2N-1} |a_\nu - b_\nu| \qquad \textit{uniformly on } \mathcal{E}.$$

Proof. By the normalization-factor in (3.31) and by (2.9), (3.3) we have

$$\Theta_n = \frac{1}{2}\Big[L(e_n + \tilde{\mathcal{T}}^{[n]}_N - \mathcal{T}_N)\, \tilde{\Phi}_{n+N} + L f_n \tilde{\Phi}^*_{n+N} + \frac{z^N}{2}(L^2 - (\tilde{L}^{[n]})^2)\, \tilde{\Phi}_n \Big].$$

From the recurrence-relation (3.2) one can obtain by an induction argument that for all $j \in \mathbb{N}_0$,

$$\left.\begin{array}{l} |\tilde{P}_N^{(n+N)}(e^{i\varphi}) - \tilde{P}_N^{(n)}(e^{i\varphi})| \\ |\tilde{\Omega}_N^{(n+N)}(e^{i\varphi}) - \tilde{\Omega}_N^{(n)}(e^{i\varphi})| \end{array}\right\}$$

$$\leqslant 2^N \sum_{v=n}^{n+N-1} |b_{v+N} - b_v|$$

$$\leqslant 2^N \sum_{v=n}^{n+2N-1} |a_v - b_v| \qquad \text{uniformly on } \mathscr{E}.$$

The same estimates hold true for

$$|\tilde{P}_N^{(n+N)*} - \tilde{P}_N^{(n)*}|,\ |\tilde{\Omega}_N^{(n+N)*} - \tilde{\Omega}_N^{(n)*}|,\ |\tilde{P}_N^{(n)[*]} - P_N^{[*]}| \text{ and } |\tilde{\Omega}_N^{(n)[*]} - \Omega_N^{[*]}|$$

(compare the $^{[*]}$-notation in (2.36)). Further, from condition (3.12) it follows for all $n \in \mathbb{N}_0$ by standard analysis that

$$\left| \prod_{v=n}^{n+N-1} (1 - |a_v|^2) - \prod_{v=n}^{n+N-1} (1 - |b_v|^2) \right| \leqslant \text{const} \cdot \sum_{v=n}^{n+N-1} |a_v - b_v|.$$

By combining these estimates and using Lemma 3.1, the assertion can be obtained. ∎

We now can give explicit representations of the perturbed orthogonal polynomials on the arcs Γ_{E_l}.

THEOREM 3.4. *Let $\varphi \in E_l$ and $\gamma = \gamma(\varphi)$ be given as in* (2.35). *Then there holds for all $m \in \mathbb{N}_0$ and $v \in \mathbb{N}$,*

$$\tilde{\Phi}_{m+(v+1)N}(e^{i\varphi}) = \frac{iLe^{i\frac{(v+1)N}{2}\varphi}}{\sqrt{R(e^{i\varphi})}\,\mathscr{U}_{N-l}(e^{i\varphi})}$$

$$\times \left(\sin(v+1)\gamma \cdot \tilde{\Phi}_{m+N}(e^{i\varphi}) - e^{i\frac{N}{2}\varphi} \sin v\gamma \cdot \tilde{\Phi}_m(e^{i\varphi}) \right.$$

$$\left. + \frac{4}{L^2} \sum_{j=0}^{v-1} e^{-i\frac{(j+1)N}{2}\varphi} \sin(v-j)\gamma \cdot \Theta_{m+jN}(e^{i\varphi}) \right), \qquad (3.32)$$

where the functions Θ_{m+jN} are given as in (3.31).

Proof. From Lemma 3.2 and (2.9) one obtains

$$\frac{2\sqrt{R}\,\mathcal{U}_{N-l}}{L}\tilde{\Phi}_{m+(v+1)N}$$

$$=\left(\tilde{\Phi}_{m+(v+2)N}-\frac{\mathcal{T}_N-\sqrt{R}\,\mathcal{U}_{N-l}}{L}\tilde{\Phi}_{m+(v+1)N}\right)$$

$$-\left(\tilde{\Phi}_{m+(v+2)N}-\frac{\mathcal{T}_N+\sqrt{R}\,\mathcal{U}_{N-l}}{L}\tilde{\Phi}_{m+(v+1)N}\right)$$

$$=\left[\frac{\mathcal{T}_N+\sqrt{R}\,\mathcal{U}_{N-l}}{L}\right]^{v+1}\left(\tilde{\Phi}_{m+N}-\frac{\mathcal{T}_N-\sqrt{R}\,\mathcal{U}_{N-l}}{L}\tilde{\Phi}_m\right)$$

$$-\left[\frac{\mathcal{T}_N-\sqrt{R}\,\mathcal{U}_{N-l}}{L}\right]^{v+1}\left(\tilde{\Phi}_{m+N}-\frac{\mathcal{T}_N+\sqrt{R}\,\mathcal{U}_{N-l}}{L}\tilde{\Phi}_m\right)$$

$$+\frac{4}{L^2}\sum_{j=0}^{v-1}\left\{\left[\frac{\mathcal{T}_N+\sqrt{R}\,\mathcal{U}_{N-l}}{L}\right]^{v-j}-\left[\frac{\mathcal{T}_N-\sqrt{R}\,\mathcal{U}_{N-l}}{L}\right]^{v-j}\right\}\cdot\Theta_{m+jN}.$$

If we substitute (2.35) in this equation we obtain (3.32). ∎

REMARK 3.3. (a) *From* (3.32) *we immediately obtain another representation of the undisturbed "periodic" orthonormal polynomials, compare Theorem* 2.1(b),

$$\Phi_{m+(v+1)N}(e^{i\varphi})$$

$$=\frac{iLe^{i\frac{(v+1)N}{2}\varphi}}{\sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})}$$

$$\times(\sin(v+1)\gamma\cdot\Phi_{m+N}(e^{i\varphi})-e^{i\frac{N}{2}\varphi}\sin v\gamma\cdot\Phi_m(e^{i\varphi})). \tag{3.33}$$

(b) *Concerning the representation in* (3.32) *let us note that* (*compare the proof of Corollary* 2.2)

$$\left|\frac{\sin v\gamma}{\sqrt{R(e^{i\varphi})}\,\mathcal{U}_{N-l}(e^{i\varphi})}\right|\leqslant\left|\frac{\sin v\gamma(\varphi_j)}{\sqrt{R(e^{i\varphi_j})}\,\mathcal{U}_{N-l}(e^{i\varphi_j})}\right|=\frac{v}{L},\qquad v\in\mathbb{N},$$

where the φ_j's are the boundary-points of the set E_l. Note that by Corollary 2.2 *and Lemma* 3.1 *the Φ_n's and $\tilde{\Phi}_n$'s are uniformly bounded at the zeros of $\mathcal{U}_{N-l}$, which are all lying in* $\mathrm{Int}(E_l)$ *by* [18, *Theorem* 3.1(c)], *while this does not hold at the boundary-points of Γ_{E_l}.*

We now suppose that (3.12) is fulfilled and define, $m \in \{0, ..., N-1\}$,

$$\kappa_{1,\nu}^{(m)}(\varphi) := \frac{4}{L^2} \sum_{j=0}^{\nu-1} e^{-i \frac{(j+1)N}{2} \varphi} e^{-ij\gamma(\varphi)} \Theta_{m+jN}(e^{i\varphi})$$
$$\kappa_{2,\nu}^{(m)}(\varphi) := \frac{4}{L^2} \sum_{j=0}^{\nu-1} e^{-i \frac{(j+1)N}{2} \varphi} e^{ij\gamma(\varphi)} \Theta_{m+jN}(e^{i\varphi}). \tag{3.34}$$

Concerning the sum in (3.32) we can write

$$\frac{8i}{L^2} \sum_{j=0}^{\nu-1} e^{-i \frac{(j+1)N}{2} \varphi} \sin(\nu-j)\gamma \cdot \Theta_{m+jN}(e^{i\varphi}) = e^{i\nu\gamma} \kappa_{1,\nu}^{(m)}(\varphi) - e^{-i\nu\gamma} \kappa_{2,\nu}^{(m)}(\varphi).$$

Now by Lemma 3.3 there exist functions $\kappa_1^{(m)}$ and $\kappa_2^{(m)}$ such that

$$\kappa_{1,\nu}^{(m)}(\varphi) \xrightarrow[\nu \to \infty]{} \kappa_1^{(m)}(\varphi) := \frac{4}{L^2} \sum_{j=0}^{\infty} e^{-i \frac{(j+1)N}{2} \varphi} e^{-ij\gamma(\varphi)} \Theta_{m+jN}(e^{i\varphi})$$
$$\kappa_{2,\nu}^{(m)}(\varphi) \xrightarrow[\nu \to \infty]{} \kappa_2^{(m)}(\varphi) := \frac{4}{L^2} \sum_{j=0}^{\infty} e^{-i \frac{(j+1)N}{2} \varphi} e^{ij\gamma(\varphi)} \Theta_{m+jN}(e^{i\varphi})$$

uniformly compact on E_l and

$$|\kappa_{s,\nu}^{(m)}(\varphi) - \kappa_s^{(m)}(\varphi)|$$
$$= \mathcal{O}\left(\sum_{j=m+\nu N}^{\infty} |a_j - b_j| \right), \qquad s = 1, 2, \text{ uniformly compact on } E_l.$$

Finally we define on $\mathrm{Int}(E_l)$

$$S_1^{(m)}(\varphi) := e^{i\gamma(\varphi)} \tilde{\Phi}_{m+N}(e^{i\varphi}) - e^{i\frac{N}{2}\varphi} \tilde{\Phi}_m(e^{i\varphi}) + \kappa_1^{(m)}(\varphi)$$
$$S_2^{(m)}(\varphi) := e^{-i\gamma(\varphi)} \tilde{\Phi}_{m+N}(e^{i\varphi}) - e^{i\frac{N}{2}\varphi} \tilde{\Phi}_m(e^{i\varphi}) + \kappa_2^{(m)}(\varphi). \tag{3.35}$$

Then by Theorem 3.4 and the above considerations we have

COROLLARY 3.1. *Suppose that* $\mathscr{E}$ *is a compact subset of* $\mathrm{Int}(E_l)$ *and let* $\gamma=\gamma(\varphi)$ *be defined as in* (2.35). *Suppose further that assumption* (3.12) *is fulfilled. Then*

$$\left| \mathscr{U}_{N-l}(e^{i\varphi}) \frac{\tilde{\Phi}_{m+\nu N}(e^{i\varphi})}{e^{i\frac{\nu N}{2}\varphi}} - \frac{L}{2\sqrt{R(e^{i\varphi})}} \{e^{i(\nu-1)\gamma} S_1^{(m)}(\varphi) - e^{-i(\nu-1)\gamma} S_2^{(m)}(\varphi)\} \right|$$
$$= \mathscr{O}\left(\sum_{j=m+(\nu-1)N}^{\infty} |a_j - b_j| \right) \tag{3.36}$$

uniformly on $\mathscr{E}$.

4. THE ORTHOGONALITY MEASURE OF THE PERTURBED POLYNOMIALS

By (3.1) there exists a measure $\tilde{\sigma}$, normed by $1/2\pi \int_0^{2\pi} d\tilde{\sigma}(\varphi)=1$, with respect to which the $\tilde{P}_n$'s are orthogonal, i.e.,

$$\int_0^{2\pi} e^{-ij\varphi} \tilde{P}_n(e^{i\varphi})\, d\tilde{\sigma}(\varphi) = 0 \qquad \text{for} \quad j=0, ..., n-1. \tag{4.1}$$

From [18, Theorem 4.4] one obtains, compare also (2.5), that the "periodic" polynomials $\{\tilde{P}_\nu^{[n]}\}_{\nu\in\mathbb{N}_0}$, given as in (3.29), are orthogonal with respect to the measure

$$d\tilde{\sigma}^{[n]}(\varphi) := \tilde{f}^{[n]}(\varphi)\, d\varphi - 2\pi \sum_{j=1}^{\tilde{p}^{[n]}} \tilde{\mu}_j^{[n]} e^{-i\varphi}\, \delta(e^{i\varphi} - e^{i\tilde{\xi}_j^{[n]}})\, d\varphi \tag{4.2}$$

with

$$\tilde{f}^{[n]}(\varphi) := \begin{cases} \left| \dfrac{\sqrt{\tilde{\mathfrak{R}}^{[n]}(e^{i\varphi})}}{\tilde{\mathfrak{A}}^{[n]}(e^{i\varphi})} \right|, & \varphi \in \tilde{E}_N^{[n]} \\ 0, & \varphi \notin \tilde{E}_N^{[n]} \end{cases} \tag{4.3}$$

where for n sufficiently large polynomials $\tilde{\mathfrak{R}}^{[n]}$ and $\tilde{\mathfrak{A}}^{[n]}$ are given as in (3.23) resp. (3.25) and where

$$\tilde{E}_N^{[n]} := \{\varphi \in [0, 2\pi] : e^{-iN\varphi} \tilde{\mathfrak{R}}^{[n]}(e^{i\varphi}) \leqslant 0\} =: \bigcup_{j=0}^{N} [\tilde{\varphi}_{2j-1}^{[n]}, \tilde{\varphi}_{2j}^{[n]}], \tag{4.4}$$

i.e., the $e^{\tilde{\varphi}_j^{[n]}}$'s, $j=1, ..., 2N$, are the zeros of $\tilde{\mathfrak{R}}^{[n]}$. Further $\tilde{\xi}_j^{[n]} \notin \tilde{E}_N^{[n]}$, $j=1, ..., \tilde{p}^{[n]}$, and the $e^{i\tilde{\xi}_j^{[n]}}$'s are zeros of $\tilde{\mathfrak{A}}^{[n]}$ on the unit circle and

$$\tilde{\mu}_j^{[n]} := \frac{\sqrt{\tilde{\mathfrak{R}}^{[n]}(e^{i\tilde{\xi}_j^{[n]}})}}{\tilde{\mathfrak{A}}_j^{[n]}(e^{i\tilde{\xi}_j^{[n]}})}, \qquad \tilde{\mathfrak{A}}_j^{[n]}(z) := \frac{\tilde{\mathfrak{A}}^{[n]}(z)}{z - e^{i\tilde{\xi}_j^{[n]}}}. \tag{4.5}$$

By the fact that the first $n+N$ reflection coefficients corresponding to $\tilde{\sigma}$ and $\tilde{\sigma}^{[n]}$ coincide, note (3.29), it follows from [7, Theorem 4.1 resp. formula (3.2)] that

$$\int_0^{2\pi} e^{-ij\varphi}\, d\tilde{\sigma}^{[n]}(\varphi) = \int_0^{2\pi} e^{-ij\varphi}\, d\tilde{\sigma}(\varphi) \qquad \text{for} \quad j=0, ..., n+N. \tag{4.6}$$

Since the sequence $\{\tilde{\sigma}^{[n]}\}_{n \in \mathbb{N}_0}$ is uniformly bounded (note that the $\tilde{\sigma}^{[n]}$'s are nondecreasing and $\tilde{\sigma}^{[n]}([0, 2\pi]) = \int_0^{2\pi} d\tilde{\sigma}^{[n]}(\varphi) = 2\pi$), by Helly's theorem we can extract a subsequence $\{\tilde{\sigma}^{[n_\nu]}\}_{\nu \in \mathbb{N}_0}$ which converges pointwise to a limit measure (resp. distribution function) $\tilde{\sigma}^{[\infty]}$ and there holds

$$\lim_{\nu \to \infty} \int_0^{2\pi} g(\varphi)\, d\tilde{\sigma}^{[n_\nu]}(\varphi) = \int_0^{2\pi} g(\varphi)\, d\tilde{\sigma}^{[\infty]}(\varphi) \tag{4.7}$$

for every continuous function g on $[0, 2\pi]$. By applying Theorem 13.3 of [7], it follows from (4.6) and (4.7) that

$$\tilde{\sigma}^{[\infty]} \equiv \tilde{\sigma}. \tag{4.8}$$

The following theorem describes the absolute continuous part of the orthogonality measure of the perturbed polynomials $\tilde{\Phi}_n$.

THEOREM 4.1. *Let $\tilde{\sigma}$ denote the orthogonality measure of the perturbed polynomials $\tilde{\Phi}_n$, where we assume that* (3.12) *is fulfilled. Then the absolute continuous part $\tilde{f}$ of $\tilde{\sigma}$ is of the form*

$$\tilde{f}(\varphi) = \begin{cases} \left| \dfrac{\sqrt{R(e^{i\varphi})}}{\alpha(\varphi)} \right|, & \varphi \in E_l \\ 0, & \varphi \notin E_l, \end{cases} \tag{4.9}$$

where $\alpha(\varphi)$ is a differentiable function on $\operatorname{Int}(E_l)$. *Further $\tilde{\sigma}$ has no mass-points on* $\operatorname{Int}(E_l)$.

Proof. By (3.1) and Assumption 2.1 one can show by using the same techniques as Golinskii *et al.* in the proof of [9, Theorem 3] that the set of accumulation points of $\operatorname{supp}(\sigma)$ and $\operatorname{supp}(\tilde{\sigma})$ coincide; as in the sections before, σ denotes the orthogonality measure of the "periodic" polynomials Φ_n.

Hence, the absolute continuous part of $\tilde{\sigma}$ must vanish identically outside E_l. We now show that the absolute continuous parts $\tilde{f}^{[n]}$ of the measures $\tilde{\sigma}^{[n]}$, defined in (4.2) resp. (4.3), converge uniformly compact on $\mathrm{Int}(E_l)$ to a function of the form (4.9) as $n \to \infty$. Then the assertion follows by the considerations, compare especially (4.7) and (4.8), at the beginning of this section.

By applying the recurrence-relation (3.2) n-times we obtain after some straight forward calculation

$$\begin{aligned}
&\frac{1}{\sqrt{d_N}\, z^{n+\frac{N}{2}}} \tilde{\mathfrak{A}}^{[n]} \\
&= \frac{1}{z^{n+\frac{N}{2}}} (\tilde{\Phi}_n \tilde{\Phi}^*_{n+N} - \tilde{\Phi}^*_n \tilde{\Phi}_{n+N}) \\
&= \frac{1}{z^{\frac{N}{2}}} \lambda_n \Bigg((\tilde{\Phi}^*_N - \tilde{\Phi}_N) + \sum_{\nu=0}^{n-1} \frac{\beta_\nu}{z^{\nu+1}} \\
&\quad \times \{ z(\bar{b}_\nu b_{\nu+N} - b_\nu \bar{b}_{\nu+N}) \tilde{\Phi}^*_\nu \tilde{\Phi}_{\nu+N} + z^2 (b_\nu - b_{\nu+N}) \tilde{\Phi}_\nu \tilde{\Phi}_{\nu+N} \\
&\quad - (\bar{b}_\nu - \bar{b}_{\nu+N}) \tilde{\Phi}^*_\nu \tilde{\Phi}^*_{\nu+N} \} \Bigg),
\end{aligned} \tag{4.10}$$

where

$$\lambda_n = \prod_{j=0}^{n-1} \left(\frac{1 - b_j \bar{b}_{j+N}}{1 - |a_j|^2} \right), \qquad \beta_\nu = \frac{1}{1 - |a_\nu|^2} \prod_{j=0}^{\nu} \left(\frac{1 - |a_j|^2}{1 - b_j \bar{b}_{j+N}} \right).$$

From (3.9), (3.12), and Lemma 3.1 it follows that the sum on the right-hand side of the above equation converges absolutely and uniformly compact on $\mathrm{Int}(E_l)$. By the fact that $ie^{-i(n+N/2)\varphi} \tilde{\mathfrak{A}}^{[n]}(e^{i\varphi})$ is a real trigonometric polynomial we have

$$\frac{ie^{-i(n+\frac{N}{2})\varphi}}{\sqrt{d_N}} \tilde{\mathfrak{A}}^{[n]}(e^{i\varphi}) \xrightarrow[n \to \infty]{} \tilde{\alpha}(\varphi) \qquad \text{uniformly compact on } \mathrm{Int}(E_l),$$

where $\tilde{\alpha}(\varphi)$ is a real differentiable function. Now recall (3.28) and the fact that $\mathcal{U}_{N-l}$ has all its zeros in $\mathrm{Int}(E_l)$ by [18, Theorem 3.1(c)]. Thus one can obtain (compare the considerations in [18, (2.13)–(2.14)]) that $\tilde{\alpha}$ can be written as

$$\tilde{\alpha}(\varphi) =: \alpha(\varphi)\, u_{N-l}(\varphi),$$

where $u_{N-l}(\varphi)$ is given as in (2.11), and (4.9) follows.

Let now φ_0 be an arbitrary point from $\operatorname{Int}(E_l)$. By Lemma 3.1 the $\tilde{\Phi}_n(e^{i\varphi_0})$'s are bounded away from zero uniformly for all n. Thus $\sum_{n=0}^{\infty} |\tilde{\Phi}_n(e^{i\varphi_0})|^2 = \infty$ and by [10, (7) and (11)] there is no mass-point at φ_0. ∎

Let us note that in general $\tilde{\sigma}$ may have an infinite number of mass-points which accumulate in (some of) the boundary-points of E_l. The following theorem shows that this is not possible if the a_n's and b_n's converge sufficiently fast to each other.

THEOREM 4.2. *Let $\tilde{\sigma}$ be the orthogonality measure of the perturbed orthogonal polynomials and let us assume that there exists a positive value $r < 1$ such that*

$$|a_n - b_n| = \mathcal{O}(r^n). \tag{4.11}$$

Then $\operatorname{supp}(\tilde{\sigma}) \backslash E_l$ *is finite, i.e., $\tilde{\sigma}$ has at most a finite number of mass-points on* $[0, 2\pi] \backslash \operatorname{Int}(E_l)$ *and no mass-point on* $\operatorname{Int}(E_l)$.

Proof. First let us recall, as already stated at the beginning of the proof of the previous theorem, that the accumulation points of $\operatorname{supp}(\sigma)$ and $\operatorname{supp}(\tilde{\sigma})$ coincide. In Theorem 4.1 we also have seen that there are no mass-points of $\tilde{\sigma}$ in $\operatorname{Int}(E_l)$. We will show that under the assumption (4.11) the zeros of $\lim_{n \to \infty} \tilde{\mathfrak{A}}^{[n]}$, $n \in \mathbb{N}_0$, from (3.25) cannot accumulate at boundary-points of Γ_{E_l}. Then taking into consideration that the mass-points of the measures $\tilde{\sigma}^{[n]}$ can only appear at points $\psi \in [0, 2\pi]$, where $\tilde{\mathfrak{A}}^{[n]}(e^{i\psi}) = 0$, the assertion follows by applying Helly's Theorem (compare the considerations from the beginning of this section).

For $\delta \in (0, 1]$, let

$$\mathcal{U}_\delta^I := \{ z \in \mathbb{C} \colon |z| \leqslant 1, |z - e^{i\varphi_j}| \leqslant \delta, j = 1, ..., 2l \}$$

and let

$$q_\delta := \sup_{z \in \mathcal{U}_\delta^I} \left| \frac{\mathcal{T}_N(z) + \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)}{L} \right|.$$

By Lemma 2.1 in conjunction with $|\mathcal{T}_N+\sqrt{R}\,\mathcal{U}_{N-l}|=L$ on Γ_{E_l} and the continuity of $\mathcal{T}_N+\sqrt{R}\,\mathcal{U}_{N-l}$ on $\mathbb{C}\backslash\Gamma_{E_l}$ we have $q_\delta>1$ and $\lim_{\delta\to 0^+}(1-\delta)/q_\delta^2=1$. Thus we can choose an $\varepsilon>0$ such that

$$r<\frac{1-\varepsilon}{q_\varepsilon^2}. \tag{4.12}$$

For abbreviation we write in the following q instead of q_ε. From [16, Lemma 2.1(c)] one obtains the following estimate uniformly on $\mathcal{U}_\varepsilon^I$:

$$\frac{|\tilde{\Phi}_n^*(z)|}{q^n}\leqslant c_1\frac{|\Phi_n^*(z)|}{q^n}+c_2\sum_{\nu=0}^{n-1}|a_\nu-b_\nu|\,q^\nu\left(\frac{|\Phi_{n-\nu}^{(\nu)*}(z)|+|\Psi_{n-\nu}^{(\nu)*}(z)|}{q^n}\right)\times\frac{|\tilde{\Phi}_\nu^*(z)|}{q^\nu};$$

here c_1 and c_2 are positive constants. With the help of Theorem 2.1, Corollary 2.2, and the maximum principle one can show that the expressions $|\Phi_n^*|/q^n$ and $(|\Phi_{n-\nu}^{(\nu)*}|+|\Psi_{n-\nu}^{(\nu)*}|)/q^n$ are uniformly bounded on $\mathcal{U}_\varepsilon^I$ (for the boundedness of $\Phi_{n-\nu}^{(\nu)*}/q^n$ and $\Psi_{n-\nu}^{(\nu)*}/q^n$ compare also the beginning of the proof of Corollary 2.2). Thus from (4.11), (4.12), and Gronwall's inequality it follows that

$$\frac{|\tilde{\Phi}_n^*(z)|}{q^n}\leqslant c_3\cdot\exp\left(c_4\sum_{\nu=0}^{\infty}|a_\nu-b_\nu|\cdot q^\nu\right)\leqslant K<\infty \qquad \text{uniformly on } \mathcal{U}_\varepsilon^I, \tag{4.13}$$

where c_3, c_4 and K are again positive constants.

Next we show that the functions $\tilde{\mathfrak{A}}^{[n]}/\sqrt{d_N}\,z^{n+N/2}$, $n\in\mathbb{N}_0$, from (4.10) converge uniformly on $\mathcal{U}_\varepsilon^I$ as $n\to\infty$ to an analytic function. The λ_n's in (4.10) are convergent and thus it sufficies to show that the sum on the right-hand side of (4.10) is uniformly convergent on $\mathcal{U}_\varepsilon^I$. Since $\{\beta_\nu\}_{\nu\in\mathbb{N}_0}$ is a bounded sequence it follows from (4.13) and (3.9) that this sum is absolutely and uniformly bounded on $\mathcal{U}_\varepsilon^I$ by

$$\text{const}\cdot\sum_{\nu=0}^{\infty}\frac{|a_\nu-b_\nu|\,q^{2\nu+N}}{(1-\varepsilon)^{\nu+1}}, \tag{4.14}$$

where we have used the fact that both $|\bar{b}_\nu b_{\nu+N}-b_\nu\bar{b}_{\nu+N}|$ and $|b_\nu-b_{\nu+N}|$ are less than or equal to $2(|a_\nu-b_\nu|+|a_{\nu+N}-b_{\nu+N}|)$ and that $|z|\geqslant 1-\varepsilon$ on $\mathcal{U}_\varepsilon^I$. By (4.11) and (4.12) the sum in (4.14) is convergent and hence

$$g(z):=\lim_{n\to\infty}\frac{1}{\sqrt{d_N}\,z^{n+\frac{N}{2}}}\tilde{\mathfrak{A}}^{[n]}(z) \qquad \text{is analytic on } \mathcal{U}_\varepsilon^I. \tag{4.15}$$

From

$$\frac{1}{z^{n+N/2}} \tilde{\mathfrak{A}}^{[n]}(z) = -\overline{\frac{1}{y^{n+N/2}} \tilde{\mathfrak{A}}^{[n]}(y)}, \qquad y = 1/\bar{z},$$

it follows that the limit-expression in (4.15) even exists on $\mathcal{U}_\varepsilon := \{z \in \mathbb{C}: |z - e^{i\varphi_j}| < \varepsilon, \ j = 1, ..., 2l\}$ and that g is analytic on $\mathcal{U}_\varepsilon$. Note that g cannot vanish identically on $\mathcal{U}_\varepsilon$, compare the proof of Theorem 4.1, and thus g has at most a finite number of zeros in the interior of $\mathcal{U}_\varepsilon$. If $g(e^{i\varphi_j}) \neq 0$ for all $j = 1, ..., 2l$, then by the uniform convergence (4.15) on $\mathcal{U}_\varepsilon$ the zeros of all the $\tilde{\mathfrak{A}}^{[n]}$'s, cannot accumulate at $e^{i\varphi_j}$ and the theorem is proven by the considerations from the beginning of this proof. If $g(e^{i\varphi_j}) = 0$ for a $j \in \{0, ..., 2l\}$ then consider a fixed neighbourhood $\mathcal{U}(\varphi_j)$ of φ_j with $g(e^{i\varphi}) \neq 0$ for $\varphi \in \mathcal{U}(\varphi_j) \setminus \{\varphi_j\}$. Again by the uniform convergence it follows that for every $\eta > 0$ there exists an index n_0 such that all the zeros of the $\tilde{\mathfrak{A}}^{[n]}$'s, $n \geqslant n_0$, which generate mass-points of $\tilde{\sigma}^{[n]}$ and which tend to $e^{i\varphi_j}$ are contained in $\{e^{i\varphi}: \varphi \in [\varphi_j - \eta, \varphi_j + \eta] \cap \mathcal{U}(\varphi_j)\}$. Hence, by (4.2), (4.8), and Helly's theorem, $\tilde{\sigma}$ cannot have mass-points on $\mathcal{U}(\varphi_j) \setminus [\varphi_j - \eta, \varphi_j + \eta]$. Since this holds, as mentioned, for every $\eta > 0$, the assertion follows. ▮

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